Actions of Inverse Semigroups and their Ample Groupoids

by

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Abstract

We study actions of inverse semigroups on arbitrary spaces, and relate properties of the action to topological properties of the associated groupoid of germs. Then, following the construction of Paterson in his 1999 book "Inverse Semigroups, Groupoids and their Operator Algebras", we investigate the intrinsic actions of inverse semigroups on their character spaces, and their groupoids of germs. We draw inspiration from a well-known result that characterizes sub-semigroups of the inverse semigroup of open bisections on an étale groupoid, which we call *bisection wideness*. Using Paterson's approach, we construct a number of ample groupoids associated to an inverse semigroup, these being the universal groupoid and the groupoid of ultragerms, and determine conditions under which an inverse semigroup $\mathcal{S}$ and a sub-semigroup $\mathcal{W}$ give rise to the same groupoid.

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Notation

$\mathcal{F}(E(\mathcal{S}))$ The set of filters on $E(\mathcal{S})$.

$\mathcal{F}_P(E(\mathcal{S}))$ The set of principal filters on $E(\mathcal{S})$.

Int(X) The topological interior of the set X.

$\mathrm{Iso}(\mathcal{G})$ The isotropy subgroupoid of a groupoid $\mathcal{G}$.

$\mathcal{L}(E(\mathcal{S}))$ The set of filters on $E(\mathcal{S})$.

$\mathcal{G}_0(\mathcal{S})$ The contracted universal groupoid of $\mathcal{S}$.

$\mathcal{G}_u(\mathcal{S})$ The universal groupoid of $\mathcal{S}$.

$\mathcal{G}_\infty(\mathcal{S})$ The groupoid of ultragerms of $\mathcal{S}$.

$\mathrm{Spec}(\mathcal{S})$ The collection of non-zero Boolean homomorphisms from $E(\mathcal{S})$ to $\{0,1\}$.

$\mathcal{U}(E(\mathcal{S}))$ The set of ultrafilters on $E(\mathcal{S})$.

$\widehat{E(\mathcal{S})}$ The collection of non-zero semigroup homomorphisms from $E(\mathcal{S})$ to $\{0,1\}$.

$\widehat{E_0(\mathcal{S})}$ The collection of non-zero monoid homomorphisms from $E(\mathcal{S})$ to $\{0,1\}$.

$\mathcal{I}(X)$ The inverse semigroup of partial bijections on subsets of X.

$\mathcal{I}_n$ The inverse semigroup of partial bijections on the finite n-element set.

$E(\mathcal{S})$ The idempotent sub-semigroup of the inverse semigroup $\mathcal{S}$.

Chapter 1

Introduction

1.1 Background

The connection between inverse semigroups and groupoids is an active area of research, particularly in the area of operator algebras. However, they originally evolved quite independently, and with different motivations.

Inverse semigroups were initially studied in the mid-20th century by Ehresmann [Ehr57], Wagner [Wag52] and Preston [Pre54]. They were developed as an algebraic analogue of pseudogroups, which are collections of partial homeomorphisms on open sets of a topological space.

Similarly, groupoids were first investigated by the German mathematician Heinrich Brandt in the early 20th-century. They were first introduced under the name "gruppoid", from the German term "gruppe" for group [Bra27]. As the name suggests, groupoids are a generalization of groups insofar as the multiplication may only be partially defined. Groups are generally considered models of global symmetry, and so groupoids were motivated by the desire to model partial symmetry. Alan Weinstein's article "Groupoids: Unifying Internal and External Symmetry" is a classical exposition on groupoids from the symmetry perspective.

The relevance of inverse semigroups and groupoids to one another is especially apparent when studying operator algebras, and in particular, C^*-algebras. Historically, these have been studied for their applications to quantum mechanics and the representation theory of locally compact groups, among other things. They were first introduced by name by Gelfand & Naimark [GN43] and Segal [Seg47] in the 1940s.

In 1980, Renault pioneered the groupoid approach to C^*-algebras, and since then, the study of groupoid C^*-algebras has been immensely fruitful [Ren80]. Of particular interest are étale groupoids, which are topological groupoids that are locally homeomorphic to their unit space. Some authors characterize étale groupoids as those topological groupoids whose collection of open sets forms a monoid under subset multiplication [Res07, Law23]. Lawson puts it succinctly; "... étale groupoids are those topological groupoids that have an algebraic alter ego."

Almost two decades later, in his book "Groupoids, Inverse Semigroups, and their Operator Algebras", Paterson studied the relationship between these three structures [Pat99]. In particular, he introduced the construction of a C^*-algebra from an inverse semigroup, as well as describing the "universal groupoid" of an inverse semigroup, obtained by finding the groupoid of germs of a canonical action of the inverse semigroup on its space of semicharacters. In doing so, he shows that the C^*-algebra of an inverse semigroup is homeomorphic to the groupoid C^*-algebra of its universal groupoid.

Particularly important in the study of étale groupoids are open bisections. These are open subsets of a groupoid that are homeomorphic to their range (equivalently, their source) as an open subset of the unit space. It is desirable to work with open bisections of a groupoid whenever possible, as they are nicely behaved - in particular, because the range and source maps are injective when restricted to an open bisection. A common consequence of a groupoid being étale is that its collections of open bisections form a basis for its topology. In the groupoids we introduce, many of our open bisections are also compact. If the collection of compact open bisections for a basis for a groupoid, we say that the groupoid is ample.

Appearing in work by Exel [Exe08], as well as Buss and Martinez [BM23], is a result that describes a canonical action of the collection of open bisections of an étale groupoid on its unit space, such that the groupoid of germs of this action is homeomorphic to the original groupoid. They proceed to characterize precisely when a sub-semigroup of open bisections can accomplish this via the restricted action - we refer to this characterization as being *bisection-wide*.

Theorem ([Exe08]). Let $\mathcal{G}$ be an étale groupoid, and let $\mathcal{S} \subseteq \mathrm{Bis}(\mathcal{G})$ be a sub-semigroup. Then $\mathcal{S}$ is bisection-wide if and only if $\mathcal{G}^{(0)} \rtimes \mathcal{S} \cong \mathcal{G}$.

In this thesis, we aim to generalize this result to the setting of Paterson. If $\mathcal{S}$ is an inverse semigroup, and $\mathcal{W} \subseteq \mathcal{S}$ a sub-semigroup, we characterize the circumstances under which $\mathcal{S}$ and $\mathcal{W}$ generate the same universal groupoid, by establishing a condition we call *wide* (see Definition 4.2.26).

Under the assumption that our inverse semigroups carry a Boolean structure, we carry out a similar process for an interesting subgroupoid of the universal groupoid, which we call the groupoid of ultragerms. We do so by utilizing a generalized formulation of Stone duality described by Lawson [Law23].

1.2 Outline

In Chapter 2, we provide a brief but rigorous introduction to groupoids, topological groupoids and étale groupoids. We then gain motivation by studying some examples, such as transformation groupoids and Deaconu-Renault groupoids.

In Chapter 3, we introduce inverse semigroups, and place particular focus on their natural partial order. We also briefly introduce Boolean algebras and Boolean inverse semigroups, which we use to study the groupoid of ultragerms of an inverse semigroup. The latter half of the chapter is dedicated to inverse semigroup actions and the groupoid of germs. Considerable effort is put into describing the sheaf topology on the groupoid of germs of an inverse semigroup action, and showing that it is an étale groupoid. We then provide both necessary and sufficient conditions on the inverse semigroup action such that the groupoid of germs possesses useful topological properties, such as those of being Hausdorff, effective, topologically free, topologically principal, and minimal.

Lastly, we describe the natural construction of an inverse semigroup from an arbitrary étale groupoid - that is, we consider its inverse semigroup of open bisections. We describe the canonical action of this collection of bisections on the unit space of the groupoid and its groupoid of germs. Following [Exe08], we show that any *bisection-wide* sub-semigroup of open bisections is sufficient to recover the original groupoid via the canonical action and its groupoid of germs.

In Chapter 4, we construct the universal gro upoid, as well as the groupoid of ultragerms and the contracted universal groupoid. This is done by describing both the semicharacter and filter approaches, and showing they are equivalent. We briefly touch on Paterson's $\mathcal{S}$-groupoid formulation of the universal groupoid, and explain its universality. Our wideness condition is then introduced, and we show that if $\mathcal{W}$ is a wide sub-semigroup of $\mathcal{S}$, then their universal groupoids are homeomorphic (see Theorem 4.2.29). The converse implication is shown to hold upon assuming that the idempotents of $\mathcal{W}$ and $\mathcal{S}$ are isomorphic (see Proposition 4.2.30).

We then describe the groupoid of ultragerms as a subgroupoid of the universal groupoid, in the case that our inverse semigroups are Boolean. This allows us to apply techniques of Stone duality to show that $\mathcal{W}$ and $\mathcal{S}$ produce the same groupoid of ultragerms if and only if they are isomorphic as inverse semigroups (see Theorem 4.3.10).

In Chapter 5, we finish by applying some of our techniques and results to various settings, such as Paterson's results on the C^*-algebras of inverse semigroups and their universal groupoids, as well as groups with a zero adjoined. We then discuss possible future lines of enquiry such as the existence and uniqueness of a minimal wide sub-semigroup, and the characterization of a wide subsemigroup of a graph inverse semigroup (as in [Pat02]).

Chapter 2

Groupoids

A groupoid is an algebraic structure that is often viewed as a generalization of a group. While groups are oftentimes used to model symmetries, groupoids are instead utilized to model *partial* symmetries - similarly, groups naturally model automorphisms of objects, while groupoids are more suited to modelling collections of morphisms between distinct objects. The study of groupoids was initiated in 1927 by Brandt [Bra27] under the name “gruppoid”. Since then, they have found relevance in many areas of modern mathematics, including category theory, homotopy theory, functional analysis, algebra and topology. A 1996 article by Weinstein [Wei96] provides an excellent coverage of the historical background and relevance of groupoids as models of symmetry.

There exists a particular sub-class of topological groupoids, known as étale groupoids. This terminology is thought to have originated from the French verb “étaler”, which means to spread out. As described by Sims, “...[étale groupoids] are the analogue, in the groupoid world, of discrete groups.” [Sim18, Section 2.4] Étale groupoids have garnered particular interest among mathematicians interested in C^*-algebras. Locally compact groupoids have been found to model many important classes of C^*-algebras, and in particular, étale groupoids are notable for reducing much of the technical prelude to these constructions while still maintaining their efficacy, giving rise to those such as Cuntz-Krieger algebras, graph algebras, and many more [Ren80, KPRR97].

2.1 Preliminaries

Briefly, a groupoid is a group such that the binary operation is only partially defined (see Definition 2.1.1 below). To some, the category-theoretic definition is more natural - a groupoid is a small category such that every arrow is an isomorphism. The reader is referred to [Sim18, Chapter 2] for an excellent introductory coverage of étale groupoids, and the associated technical details.

In the following, we establish some basic algebraic properties of groupoids. Beyond this section, we employ these without comment.

Definition 2.1.1. A *groupoid* is a set $\mathcal{G}$ along with a set of composable pairs $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$, a multiplication operation $(\alpha, \beta) \mapsto \alpha\beta$, and an inverse operation $\gamma \mapsto \gamma^{-1}$, such that the following hold.

(G1) For all $\gamma \in \mathcal{G}$, $(\gamma^{-1})^{-1} = \gamma$.

(G2) If (α, β) and (β, γ) are in $\mathcal{G}^{(2)}$, then $(\alpha\beta, \gamma)$ and $(\alpha, \beta\gamma)$ are in $\mathcal{G}^{(2)}$, and furthermore, $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

(G3) For each $\gamma \in \mathcal{G}$ one has $(\gamma, \gamma^{-1}) \in \mathcal{G}^{(2)}$, and if $(\alpha, \beta) \in \mathcal{G}^{(2)}$, then

$$\alpha^{-1}(\alpha\beta) = \beta \quad \text{and} \quad (\alpha\beta)\beta^{-1} = \alpha.$$

In general, a groupoid doesn't admit a unique identity, but has a distinguished subset of elements that behave like "local" identities, and are defined analogously as those elements equal to the product $\gamma\gamma^{-1}$ (equivalently, $\gamma^{-1}\gamma$) for some $\gamma \in \mathcal{G}$. This set is called the *unit space* of $\mathcal{G}$, and is denoted by $\mathcal{G}^{(0)}$. We define the *source* and *range* maps $d, r : \mathcal{G} \to \mathcal{G}^{(0)}$, respectively, by

$$d(\gamma) = \gamma^{-1}\gamma \quad \text{and} \quad r(\gamma) = \gamma\gamma^{-1}.$$

If $x \in \mathcal{G}^{(0)}$, and $d(\gamma) = x$ (respectively, $r(\gamma) = x$) then one has $x\gamma = \gamma$ (respectively, $\gamma x = \gamma$). The following lemma establishes the uniqueness of inverses.

Lemma 2.1.2. *If $\gamma \in \mathcal{G}$ and α, β are both inverses of γ, then $\alpha = \beta$.*

Proof. Since α and β are inverses of γ, we have

$$\alpha(\gamma\beta) = \beta \quad \text{and} \quad (\alpha\gamma)\beta = \alpha,$$

by property (G3). The associativity of multiplication given by (G2) then implies $\alpha = \beta$. □

Henceforth, for $\gamma \in \mathcal{G}$, we may speak of *the* inverse of γ, and write γ^{-1} without ambiguity.

We state the following lemma (see [Sim18, Lemma 2.1.4]).

Lemma 2.1.3. *If $\alpha, \beta \in \mathcal{G}$, then $(\alpha, \beta) \in \mathcal{G}^{(2)}$ if and only if $r(\beta) = d(\alpha)$.*

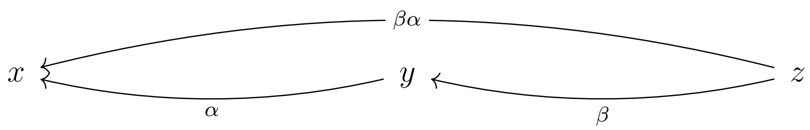


Figure 2.1: Composition of morphisms in $\mathcal{G}$.

Lemma 2.1.4. *If $(\alpha, \beta) \in \mathcal{G}^{(2)}$, then $r(\alpha\beta) = r(\alpha)$ and $d(\alpha\beta) = d(\beta)$.*

Proof. Since $(\alpha, \beta) \in \mathcal{G}^{(2)}$ and $(\alpha, \alpha^{-1}), (\alpha^{-1}, \alpha) \in \mathcal{G}^{(2)}$, two uses of (G2) gives $(\alpha\alpha^{-1}, \alpha\beta) = (r(\alpha), \alpha\beta) \in \mathcal{G}^{(2)}$. Applying both (G2) and (G3), we have

$$r(\alpha)(\alpha\beta) = (r(\alpha)\alpha)\beta = \alpha\beta = r(\alpha\beta)\alpha\beta.$$

We can then multiply each side on the right by $(\alpha\beta)^{-1}$, and again by (G3) we have $r(\alpha\beta) = r(\alpha)$. A similar argument shows that $d(\alpha\beta) = d(\beta)$. □

Lemma 2.1.5. *Let* $(\alpha, \beta) \in \mathcal{G}^{(2)}$. *Then* $(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$.

Proof. Suppose that $(\alpha, \beta) \in \mathcal{G}^{(2)}$. We have that

$$r(\alpha^{-1}) = d(\alpha) = r(\beta) = d(\beta^{-1}),$$

and so $(\beta^{-1}, \alpha^{-1}) \in \mathcal{G}^{(2)}$ by Lemma 2.1.3. Furthermore, an application of Lemma 2.1.4 gives us

$$r(\beta^{-1}\alpha^{-1}) = d(\beta) = d(\alpha\beta),$$

as $(\alpha, \beta) \in \mathcal{G}^{(2)}$. Hence, $(\beta^{-1}\alpha^{-1}, \alpha\beta) \in \mathcal{G}^{(2)}$. Similarly, one has

$$d(\alpha\beta\beta^{-1}) = d(\beta^{-1}) = r(\alpha^{-1}),$$

and so $(\alpha\beta\beta^{-1}, \alpha^{-1}) \in \mathcal{G}^{(2)}$, and $(\alpha\beta\beta^{-1})\alpha^{-1} = (\alpha\beta)\beta^{-1}\alpha^{-1}$. Then, using (G3) and Lemma 2.1.4, we have

$$(\alpha\beta)(\alpha\beta)^{-1} = r(\alpha\beta) = r(\alpha) = \alpha\alpha^{-1} = (\alpha\beta)\beta^{-1}\alpha.$$

Uniqueness of inverses (Lemma 2.1.2) then implies that $(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$ is the unique element satisfying the above equation. □

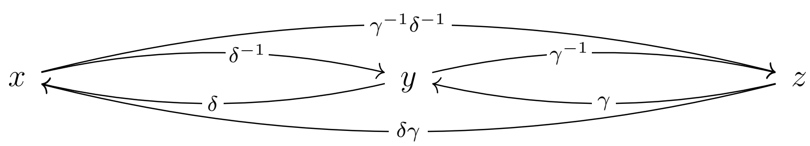


Figure 2.2: Composition of morphisms and their inverses.

Lemma 2.1.6. *If* $x \in \mathcal{G}^{(0)}$ *is a unit, then* $r(x) = d(x) = x$.

Proof. Suppose $x \in \mathcal{G}^{(0)}$ is a unit. Then $x = \gamma^{-1}\gamma$ for some $\gamma \in \mathcal{G}$. Applying Lemma 2.1.5 along with (G3), we have

$$d(\gamma^{-1}\gamma) = (\gamma^{-1}\gamma)^{-1}(\gamma^{-1}\gamma) = \gamma^{-1}\gamma\gamma^{-1}\gamma = \gamma^{-1}\gamma.$$

Similarly,

$$r(\gamma^{-1}\gamma) = (\gamma^{-1}\gamma)(\gamma^{-1}\gamma)^{-1} = \gamma^{-1}\gamma\gamma^{-1}\gamma = \gamma^{-1}\gamma.$$

□

For all $x \in \mathcal{G}^{(0)}$, we define

$$\mathcal{G}^x := \{\gamma \in \mathcal{G} : r(\gamma) = x\} \quad \text{and} \quad \mathcal{G}_x := \{\gamma \in \mathcal{G} : d(\gamma) = x\},$$

as well as $\mathcal{G}_x^x = \mathcal{G}^x \cap \mathcal{G}_x$.[1] For any unit $x \in \mathcal{G}^{(0)}$, the groupoid axioms imply that the set $\mathcal{G}_x^x$ is a group with respect to the multiplication on $\mathcal{G}$, and with x as identity. The set of all elements $\gamma \in \mathcal{G}$ such that $r(\gamma) = d(\gamma)$ is called the isotropy subgroupoid of $\mathcal{G}$, and is denoted $\mathrm{Iso}(\mathcal{G})$. One can check that $\mathrm{Iso}(\mathcal{G}) = \bigcup_{x\in\mathcal{G}^{(0)}} \mathcal{G}_x^x$, from which it is evident that $\mathrm{Iso}(\mathcal{G})$ is a union of groups - often called a *group bundle*. We will make extensive use of this set. In particular, notice that $\mathcal{G}^{(0)} \subseteq \mathrm{Iso}(\mathcal{G})$. Many important topological properties of groupoids we study are related to how much isotropy lies outside the unit space. Henceforth, we say that $x \in \mathcal{G}^{(0)}$ has non-trivial isotropy if $\mathcal{G}_x^x \neq \{x\}$.

2.1.1 Examples

Example 2.1.7. Let X be a topological space, and $\mathcal{G}$ the collection of equivalence classes of continuous paths in X, whereby two paths are equivalent if they are homotopic to one another - that is, if they can be continuously deformed into one another. Note that if two paths are homotopic, their

[1] Note that in some literature, $\mathcal{G}^x$ and $\mathcal{G}_x$ are written as $\mathcal{G}x$ and $x\mathcal{G}$, respectively.

endpoints must coincide. Then each $\gamma \in \mathcal{G}$ is an equivalence class of continuous paths from $d(\gamma)$ to $r(\gamma)$ in X, where $d(\gamma)$ is the starting point of the path, and $r(\gamma)$ is the end of the path.

Multiplication in the groupoid is given by composition of paths, whereby two paths are composable if and only if the endpoint of the first path coincides with the starting point of the second path. This is well-defined with regards to the equivalence classes, since two paths must share endpoints in order to be homotopic to one another. The inverse of a path is given by the same path travelling in the opposite direction. The unit space consists of continuous paths composed with their inverse, from which it follows that $\mathcal{G}^{(0)}$ coincides with the collection of all starting points and ending points of paths, which is just X. Hence, $\mathcal{G}$ is the *fundamental groupoid* of X. One can check that, for any given $x \in X$, the fundamental group of X based at x is the group $\mathcal{G}_x^x$, as isotropy in the groupoid corresponds to equivalence classes of continuous loops in X.

Example 2.1.8. Suppose R is an equivalence relation on a set X - that is, a subset $R \subseteq X \times X$ satisfying the following axioms of an equivalence relation.

(i) For all $a \in X$, we have $(a, a) \in R$. (Reflexivity)

(ii) If $(a, b) \in R$ then $(b, a) \in R$. (Symmetry)

(iii) If $(a, b), (b, c) \in R$ then $(a, c) \in R$. (Transitivity)

We define a groupoid $\mathcal{G} := R$. The composable pairs are given by $\mathcal{G}^{(2)} := \{(a, b), (b, c) : (a, b), (b, c) \in R\}$. Transitivity means we can naturally define multiplication as $(a, b)(b, c) := (a, c)$, and symmetry gives rise to inversion being defined as $(a, b)^{-1} := (b, a)$. It follows from reflexivity that we can take the unit space to be the set $\{(a, a) : a \in X\}$.

We identify $\mathcal{G}^{(0)}$ with X via the correspondence $x \mapsto (x, x)$. For $(x, y) \in R$, one has $d(x, y) = y$ and $r(x, y) = x$. We can see that $\mathrm{Iso}(\mathcal{G}) = \{(x, x) : x \in$

$X\}$, and so $\mathcal{G}$ has no non-trivial isotropy - that is, $\mathrm{Iso}(\mathcal{G}) = \mathcal{G}^{(0)}$. Furthermore, $\mathcal{G}$ is always a principal groupoid - that is, there exist no non-trivial isotropy groups.

Notice that each axiom of the equivalence relation corresponds naturally to the general definitions of the groupoid operations. It is an interesting fact that equivalence relations and principal groupoids are algebraically the same mathematical objects [Sim18, Lemma 2.1.14].

Both these preceding examples illustrate how the groupoid structure naturally underlies numerous other mathematical objects. Further examples include group actions (see Example 2.2.5) and the Deaconu-Renault groupoids (see Example 2.2.6), both of which we introduce in the next section.

2.2 Étale Groupoids

Before we introduce the notion of an étale groupoid, we must first discuss topological groupoids. Constructing a topological groupoid amounts to endowing a groupoid with a topology that is compatible with its groupoid structure.

Let X, Y be topological spaces. Throughout, by *homeomorphism* we mean a continuous bijection $f : X \to Y$ with a continuous inverse, and by *local homeomorphism* we mean a mapping f such that for every point $x \in X$, there exists some open neighbourhood U containing x such that $f|_U$ is a homeomorphism, and $f(U)$ is open in Y. We denote by $\mathrm{Int}(U)$ the topological interior of the set U, and by ∂U the boundary of U.

Whenever $\mathcal{G}$ is endowed with a topology, we will tacitly assume that $\mathcal{G}^{(2)}$ is equipped with the product topology inherited from the product space $\mathcal{G} \times \mathcal{G}$, and $\mathcal{G}^{(0)}$ has the subspace topology with respect to $\mathcal{G}$.

2.2.1 Topological Groupoids

Let $\mathcal{G}$ be a groupoid, and τ a topology on $\mathcal{G}$. We say $(\mathcal{G}, \tau)$ is a *topological groupoid* if the multiplication and involution maps are all continuous with respect to τ.

Henceforth, we unambiguously write $\mathcal{G}$ to mean $(\mathcal{G}, \tau)$ and omit mention of the specific topology. One can check that since the range and source maps are defined purely in terms of multiplication and inversion, if $\mathcal{G}$ is a topological groupoid then the range and source are both continuous.

Since this thesis puts no Hausdorff assumption on our groupoids, the following proposition from [Sim18, Lemma 2.3.2] will come in use. We will use the fact that a topological space X is Hausdorff if and only if every net in X converges to at most one point [Wil04, Theorem 13.7.b)].

Proposition 2.2.1. Let $\mathcal{G}$ be a topological groupoid. Then $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}$ if and only if $\mathcal{G}$ is Hausdorff.

Proof. Suppose $\mathcal{G}$ is Hausdorff, and let (γ_i) be a net in $\mathcal{G}^{(0)}$ such that $\gamma_i \to \gamma$ for some $\gamma \in \mathcal{G}$. Since the range and source mappings are continuous, $r(\gamma_i) \to r(\gamma) \in \mathcal{G}^{(0)}$. But each γ_i is a unit, so $r(\gamma_i) = \gamma_i \to r(\gamma)$. Since $\mathcal{G}$ is Hausdorff, this limit is unique i.e. $\gamma = r(\gamma) \in \mathcal{G}^{(0)}$. Hence, $\mathcal{G}^{(0)}$ is closed under limits and so is closed (see [Wil04, Theorem 11.7]).

Conversely, suppose $\mathcal{G}^{(0)}$ is closed. Let (γ_i) be a net in $\mathcal{G}$ with $\gamma_i \to \gamma_1$ and $\gamma_i \to \gamma_2$, for some $\gamma_1, \gamma_2 \in \mathcal{G}$. Then by continuity of multiplication and inversion, we have $\gamma_i^{-1}\gamma_i \to \gamma_1^{-1}\gamma_2$. But each $\gamma_i^{-1}\gamma_i$ is a unit (Lemma 2.1.2), and so $\gamma_1^{-1}\gamma_2 \in \mathcal{G}^{(0)}$, since $\mathcal{G}^{(0)}$ is closed. This implies that $\gamma_1 = \gamma_2$, and so $\mathcal{G}$ is Hausdorff. $\square$

An open set $U \subseteq \mathcal{G}$ is called an *open bisection* if $r|_U$ and $s|_U$ are homeomorphisms onto their image - in particular, they are injective - and $r(U), d(U)$ are open in $\mathcal{G}^{(0)}$. Some authors refer to open bisections as *slices*, while others (in particular, Renault) use the term *G-set* [Ren80, Definition 1.10].

There is a lack of consistency in the literature regarding the precise definition of an étale groupoid - in particular, whether it should require that $\mathcal{G}^{(0)}$ be locally compact Hausdorff. Often, this depends on the context. This thesis is only concerned with étale groupoids whose unit spaces are locally compact Hausdorff, and so we follow the formulation of Exel [Exe08].

Definition 2.2.2. A topological groupoid $\mathcal{G}$ is *étale* if $\mathcal{G}^{(0)}$ is locally compact Hausdorff, and the range map (or equivalently the source map) is a local homeomorphism.

There exist alternate, equivalent definitions for étale groupoids. Interestingly, Resende shows in [Res07, Theorem 5.18] that a groupoid $\mathcal{G}$ is étale if and only if its collection of open sets form a semigroup under pointwise multiplication, with identity being given by the open unit space. This further strengthens the notion that groupoid theory and semigroup theory are intimately linked, particularly when it comes to their connection to C^*-algebra theory.

The following facts about étale groupoids are immediate, but crucial.

Proposition 2.2.3. Let $\mathcal{G}$ be an étale groupoid. Then,

(i) The unit space $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$.

(ii) There exists a basis for $\mathcal{G}$ consisting of open bisections.

(iii) For each $x \in \mathcal{G}^{(0)}$, both $\mathcal{G}^x$ and $\mathcal{G}_x$ are discrete in the subspace topology inherited from $\mathcal{G}$.

Proof. *(i)* Since $\mathcal{G}$ is étale, for every $\gamma \in \mathcal{G}$, we can find a neighbourhood U_γ of γ such that $r|_{U_\gamma}$ is a homeomorphism, and $r(U_\gamma)$ is open in $\mathcal{G}^{(0)}$. Then $\bigcup_{\gamma\in\mathcal{G}} r(U_\gamma) = \mathcal{G}^{(0)}$ is open as an arbitrary union of open sets.

(ii) We use [Mun03, Lemma 13.1]. Suppose $\gamma \in \mathcal{G}$ is contained in the open set W. We wish to find an open bisection $B \subseteq W$ containing γ. Since r and s are local homeomorphisms, there exist open sets U_γ and V_γ containing γ

such that $r|_{U_\gamma}$ is a homeomorphism, and $s|_{V_\gamma}$ is a homeomorphism. Then, $B = U_\gamma \cap V_\gamma \cap W$ is an open bisection containing γ and contained in W.

(iii) For $x \in \mathcal{G}^{(0)}$ and $\gamma \in \mathcal{G}^x$, let U be an open bisection containing γ. Then $U \cap \mathcal{G}^x = \{\gamma\}$ since U is a bisection, and $\{\gamma\}$ is open in $\mathcal{G}^x$. A similar argument shows that if $\gamma \in \mathcal{G}_x$, then $\{\gamma\}$ is open in $\mathcal{G}_x$. Hence, $\mathcal{G}_x^x = \mathcal{G}^x \cap \mathcal{G}_x$ is discrete. □

Let $\mathcal{G}$ be an étale groupoid. We say that $\mathcal{G}$ is *effective* if $\text{Int}(\text{Iso}(\mathcal{G})) \subseteq \mathcal{G}^{(0)}$. Since one has the reverse inclusion for free, this is equivalent to requiring $\text{Int}(\text{Iso}(\mathcal{G})) = \mathcal{G}^{(0)}$ [CB20, Section 7]. We say that $\mathcal{G}$ is *topologically free* if $\text{Int}(\text{Iso}(\mathcal{G}) \setminus \mathcal{G}^{(0)}) = \emptyset$ (this is what is defined as effective in [BCFS14, Section 2]). One can think of being topologically free as a form of being *weakly effective* - the two definitions coincide if $\mathcal{G}$ is Hausdorff, and otherwise effective implies topologically free (see [CEP+19, Example 5] for an example of the converse implication failing). We also have the notion of $\mathcal{G}$ being *topologically principal*, whereby the collection $\{x \in \mathcal{G}^{(0)} : \mathcal{G}_x^x = \{x\}\}$ is dense in $\mathcal{G}^{(0)}$. See [AdCC+23, Remark 2.1] for further discussion on these relationships.

Furthermore, we say a set $U \subseteq \mathcal{G}$ is *invariant* if $d(\gamma) \in U \implies r(\gamma) \in U$ for any $\gamma \in \mathcal{G}$. If $\mathcal{G}$ admits no non-trivial open invariant subsets, then $\mathcal{G}$ is *minimal*.

Lemma 2.2.4. *Let $\mathcal{G}$ be an étale groupoid. If $\mathcal{G}$ is topologically principal, then it is topologically free.*

Proof. Suppose that $\mathcal{G}$ is not topologically free. Then $\text{Int}(\text{Iso}(\mathcal{G}) \setminus \mathcal{G}^{(0)})$ is a non-empty open set. Since the source map is open, $d(\text{Int}(\text{Iso}(\mathcal{G}) \setminus \mathcal{G}^{(0)}))$ is a non-empty open set in the unit space consisting of units with non-trivial isotropy. Hence, $\mathcal{G}$ is not topologically principal. □

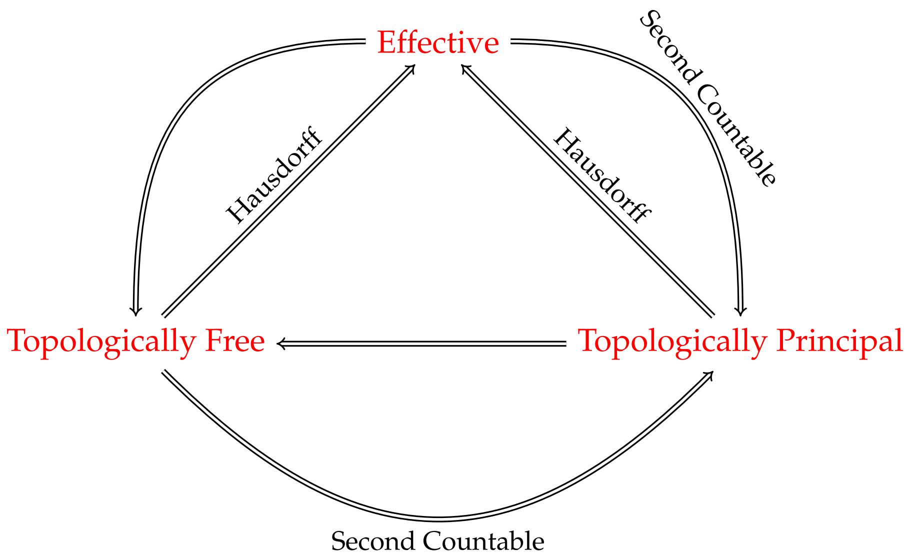


Figure 2.3: Relationships between the properties of being effective, topologically free and topologically principal.

2.2.2 Examples

Mention of the following examples can be found in [Sim18, Example 2.1.15, 2.1.16]. We note that the Deaconu-Renault groupoid was originally introduced in [Dea95], but we follow Sims' construction.

Example 2.2.5 (Transformation Groupoid)**.** The action of any group G on a set X naturally gives rise to the transformation groupoid, denoted $G \ltimes X$. This construction is discussed in detail in [CH20, Example 3.3]. We briefly describe the topology on $G \ltimes X$ in the case that G is a discrete group and X is locally compact Hausdorff, and show that $G \ltimes X$ is étale.

Under these assumptions, we can give $G \ltimes X$ the product topology. Hence, a set $U \times V \subseteq \mathcal{G}$ is a basic open set if and only if U is open in G and V is open in X. We claim that $\mathcal{G}$ is étale. Let $(\gamma, x) \in \mathcal{G}$, and consider the neighbourhood $\{\gamma\} \times X$. If $(\gamma, x), (\gamma, y) \in \{\gamma\} \times X$ with $x \neq y$, then $d(\gamma, x) = x \neq y = d(\gamma, y)$. Hence, $s|_{\{\gamma\} \times X}$ is injective.

If $\{e\} \times U \subseteq \{e\} \times X$ is open, then $r|^{-1}_{\{\gamma\}\times U}(\{e\} \times U) = \{\gamma\} \times \gamma^{-1} \cdot U$, which is open, and so $r|_{\{\gamma\}\times U}$ is continuous. The inverse mapping of $r|_{\{\gamma\}\times U}$ is $r^{-1}|_{\{\gamma\}\times U}$ which takes (e, x) to $(\gamma, \gamma^{-1} \cdot x)$. If $\{\gamma\} \times U$ is open in $\{\gamma\} \times X$, then $r|_{\{\gamma\}\times U}(\{\gamma\} \times U) = \{e\} \times \gamma \cdot U$, which is also open, and so $r^{-1}|_{\{\gamma\}\times U}$ has a continuous inverse. A similar set of arguments show that the source map is a local homeomorphism. Hence, $G \ltimes X$ is étale.

Example 2.2.6 (Deaconu-Renault Groupoid)**.** If G is an abelian group, $S \subseteq G$ a subsemigroup containing 0, and X a set, we can construct another groupoid associated to a particular action of $\mathcal{S}$ on X called the Deaconu-Renault groupoid. As above, we refer the reader to [Sim18, Example 2.1.16] for details of this construction, and we limit our discussion to the topology, and establishing that the Deaconu-Renault groupoid is étale.

Let G be a discrete abelian group, X a locally compact Hausdorff space, and $\mathcal{G}$ the associated Deaconu-Renault groupoid. There then exists a basis for $\mathcal{G}$ consisting of open sets of the form

$$Z(U, p, q, V) = \{(x, p - q, y) : x \in U, y \in V, p \cdot x = q \cdot y\},$$

and in this way is an étale groupoid. In particular, for any groupoid element $(x, p - q, y) \in \mathcal{G}$, one can choose neighbourhoods $U \ni x$ and $V \ni y$ such that $\theta_p : U \to p \cdot U$ and $\theta_q : V \to q \cdot V$ are homeomorphisms. Letting $W := p \cdot U \cap q \cdot V$, we can define $U' = \{u \in U : p \cdot u \in W\}$ and $V' = \{v \in V : q \cdot v \in W\}$, and it follows that the open set $Z(U', p, q, V')$ is an open neighbourhood of $(x, p - q, y)$ on which the range map is a homeomorphism.

Chapter 3

Inverse Semigroups

In the opening of his influential book "Groupoids, Inverse Semigroups and their Operator Algebras", Paterson explains, "In recent years, it has become increasingly clear that there are important connections relating three mathematical concepts which a priori seem to have nothing much in common. These are groupoids, inverse semigroups and operator algebras" [Pat99]. In particular, one can associate C^*-algebras to inverse semigroups via their representations, or alternatively, one can construct various topological groupoids from an inverse semigroup which often turn out to be étale, or even ample. Our interest will turn to the naturally constructed universal groupoid of an inverse semigroup (commonly called Paterson's groupoid). Paterson showed this groupoid has full and reduced C^*-algebras isomorphic to those of the original inverse semigroup.

3.1 Preliminaries

We begin this chapter by introducing some basic theory about inverse semigroups, the details of which are abundant in literature, but we follow [Law98] and [Pat99].

Definition 3.1.1. An *inverse semigroup* is a set $\mathcal{S}$ equipped with an associative binary operation $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$ such that for each $s \in \mathcal{S}$ there exists a unique $s^* \in \mathcal{S}$ (called the *inverse* of s) satisfying

$$s^*ss^* = s^* \quad \text{and} \quad ss^*s = s.$$

We say an inverse semigroup $\mathcal{S}$ is an *inverse semigroup with 0* if it contains an element 0 such that $0s = s0 = 0$ for all $s \in \mathcal{S}$. Similarly, we say that $\mathcal{S}$ is an *inverse semigroup with 1* if it contains an element 1 such that $1s = s1 = s$ for all $s \in \mathcal{S}$. General inverse semigroups need not contain either of these elements.

By *inverse sub-semigroup*, we mean a subset $\mathcal{S}'$ of $\mathcal{S}$ such that $\mathcal{S}'$ is an inverse semigroup with respect to the binary operation on $\mathcal{S}$. In this case, we write $\mathcal{S}' \leq \mathcal{S}$.

Every inverse semigroup $\mathcal{S}$ admits an inverse sub-semigroup consisting of its idempotents - that is, those elements $e \in \mathcal{S}$ such that $e^2 = e$. We denote the collection of idempotents of $\mathcal{S}$ by $E(\mathcal{S})$.

Before establishing some useful identities of inverse semigroups, we first prove a number of standard properties of idempotents.

Lemma 3.1.2. *Let $\mathcal{S}$ be an inverse semigroup, with idempotents $E(\mathcal{S})$.*

(I1) If $s \in \mathcal{S}$, then $s^*s, ss^* \in E(\mathcal{S})$.

(I2) If $e \in E(\mathcal{S})$, then $e^* = e$.

(I3) For all $e, f \in E(\mathcal{S})$, we have $ef = fe$.

Proof. *(I1)* Let $s \in \mathcal{S}$. Associativity of multiplication gives us

$$(s^*s)(s^*s) = s^*(ss^*s) = s^*s.$$

Hence, $s^*s \in E(\mathcal{S})$. Replacing s with s^* gives us $ss^* \in E(\mathcal{S})$.

(I2) Take some $e \in E(\mathcal{S})$. Since e is an idempotent, $e^n = e$ for any $n \in \mathbb{N}$. Thus, we have $eee = e$, and so $e^* = e$.

(I3) Take $e, f \in E(\mathcal{S})$. We first claim that $f(ef)^*e$ is an idempotent. We have

$$(f(ef)^*e)(f(ef)^*e) = f((ef)^*(ef)(ef)^*)e = f(ef)^*e.$$

Thus, $f(ef)^*e \in E(\mathcal{S})$. Next, we show that $f(ef)^*e$ and ef are inverses of one another. We have

$$(f(ef)^*e)(ef)(f(ef)^*e) = f(ef)^*(ee)(ff)(ef)^*e = (f(ef)^*e)(f(ef)^*e) = \\ f((ef)^*ef(ef)^*)e = (f(ef)^*e).$$

Similarly, we have

$$(ef)(f(ef)^*e)(ef) = e(ff)(ef)^*(ee)f = (ef)(ef)^*(ef) = ef.$$

Uniqueness of inverses then implies that $f(ef)^*e$ and ef are inverses of one another. But $f(ef)^*e \in E(\mathcal{S})$, and so ef is an idempotent. Now,

$$ef(fe)ef = e(ff)(ee)f = (ef)(ef) = ef,$$

due to ef being an idempotent, implying fe is an inverse of ef. The fact that inverses are unique, along with (I2), then gives $ef = fe$. □

Note that every idempotent $e \in E(\mathcal{S})$ can be written as s^*s for some $s \in \mathcal{S}$ - for instance, take $s = e$, such that $e^*e = e$.

The following results regarding inverse semigroup elements follow from the above properties of idempotents.

Lemma 3.1.3. *Let $\mathcal{S}$ be an inverse semigroup, and let $s, t \in \mathcal{S}$. Then,*

(S1) $(s^*)^* = s$.

(S2) $(st)^* = t^*s^*$.

(S3) If $e \in E(\mathcal{S})$ *then* $ses^*, s^*es \in E(\mathcal{S})$.

Proof. *(S1)* From the definition of inverses, we have $s^*ss^* = s^*$, and also $s^*(s^*)^*s^* = s^*$. Uniqueness of inverses then gives $(s^*)^* = s$.

(S2) We claim that (t^*s^*) is an inverse for st. Since s^*s and t^*t are idempotents, by (I3), we have

$$(st)(t^*s^*)(st) = stt^*s^*st = ss^*stt^*t = st.$$

Hence, t^*s^* is the inverse of st.

(S3) Let $e \in E(\mathcal{S})$ and $s \in \mathcal{S}$. We have

$$(ses^*)(ses^*) = (ss^*s)(ee)s^* = ses^*.$$

Hence, ses^* is an idempotent. An identical argument using s^* in place of s gives us that $s^*es \in E(\mathcal{S})$. □

Throughout the remainder of this thesis, we employ Lemma 3.1.2 and Lemma 3.1.3 without further justification.

3.1.1 The Natural Partial Order

For any inverse semigroup $\mathcal{S}$, there exists a natural partial order, whereby if $s, t \in \mathcal{S}$, then we say $s \leq t$ if and only if there exists an idempotent $e \in E(\mathcal{S})$ such that $s = te$ (equivalently, if there exists an idempotent $f \in E(\mathcal{S})$ such that $s = ft$). This partial order simplifies when restricted to the idempotents - in particular, if $e, f \in E(\mathcal{S})$ then $e \leq f$ if and only if $e = ef = fe$.

Let $(P, \leq)$ be a partially-ordered set. We say P is a *lattice* if P admits all pairwise meets and joins (these are also called infimums and supremums, respectively). In particular, admitting pairwise meets means if $a, b \in P$, then there exists $x \in P$ such that $x \leq a, b$, and if y is another element such that $y \leq a, b$ then $y \leq x$. Dually, admitting pairwise joins means if $a, b \in P$, then there exists $x \in P$ such that $x \geq a, b$, and if y is another element such that $y \geq a, b$ then $y \geq x$, and we write $x = a \vee b$. It follows that if P is a lattice, then P admits all finite meets and joins.

A *meet-semilattice* (respectively, a *join-semilattice*) is a partially ordered set P such that P admits all finite meets (respectively, all finite joins). In particular, all lattices are semilattices, but not all semilattices are lattices.

Given an inverse semigroup $\mathcal{S}$, the set of idempotents of $\mathcal{S}$ form a *meet-semilattice*, whereby any two elements $e, f \in E(\mathcal{S})$ admit an infimum $e \wedge f$, which in this case is given by ef. To see this, notice that $ef = (ef)f = e(ef)$, hence $ef \leq e, f$. If $g \leq e, f$ then $ge = gf = g$, which means $g(ef) = (ge)f = gf = g$, and so $g \leq ef$. Throughout, unless stated otherwise, if e, f are elements of a meet semilattice E, then ef will denote the meet $e \wedge f$.

In general, $\mathcal{S}$ itself doesn't exhibit a semilattice structure, as not all pairs of elements admit a greatest lower bound. There exists a particular class of inverse semigroups, namely *E-unitary inverse semigroups*, for which their entire structure can be characterized as a semilattice [Law98, Lemma 4.7]. In particular, we say that an inverse semigroup S is *E-unitary* if $e \in E(\mathcal{S})$ and $e \leq s$ implies $s \in E(\mathcal{S})$. We say that $\mathcal{S}$ is *E^*-unitary* if the previous statement holds for all non-zero e [Law98, p. 20].

The following lemma is from [Pat99].

Lemma 3.1.4. *[Pat99, Lemma 2.1.1] Let $s \in \mathcal{S}$ and $e \in E(\mathcal{S})$. Then $ses^* \leq ss^*$ and $s^*es \leq s^*s$.*

Proof. By inspection, one has

$$(ses^*)ss^* = se(s^*ss^*) = ses^*,$$

from which the first result follows - the second is immediate after replacing s with s^* and employing (S1). □

Suppose $\mathcal{S}$ is an inverse semigroup with 0, and let $e, f \in E(\mathcal{S})$. If there exists g such that $0 \neq g \in E(\mathcal{S})$ and $g \leq e, f$ then we write $e \Cap f$, and say that e and f *intersect*. Otherwise, we say that e and f are *disjoint*.

Example 3.1.5. The prototypical example of an inverse semigroup is as follows. Let X be any set, and denote by $\mathcal{I}(X)$ the set of partial bijections on X. Then $\mathcal{I}(X)$ is an inverse semigroup, called the *symmetric inverse semigroup on* X, where the product of elements is the composition of functions on the greatest possible domain. In particular, if $f, g \in \mathcal{I}(X)$ are partial bijections on X, then their product fg is a partial bijection with domain $g^{-1}(\mathrm{ran}(g) \cap \mathrm{dom}(f))$. Idempotents are the identity functions on their domain.

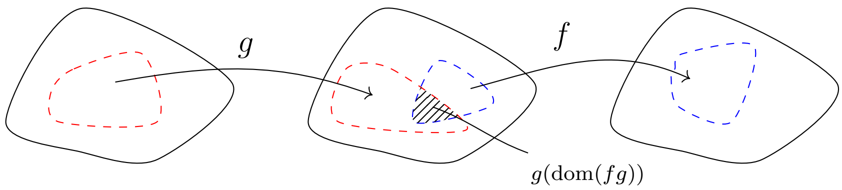


Figure 3.1: Representation of a product in a symmetric inverse semigroup.

If $f, g \in \mathcal{I}(X)$ are partial bijections, then $f \leq g$ if and only if f is equal to the restriction of g to some subset of $\mathrm{dom}(g)$. That is, if there exists an idempotent $e \in E(\mathcal{I}(X))$ such that $f = g \circ e$. If e, f are idempotents of $\mathcal{I}(X)$ then the partial order simply becomes set inclusion, such that $e \leq f$ if and only if $\mathrm{dom}(e) \subseteq \mathrm{dom}(f)$.

3.1.2 Semigroup Homomorphisms

As one would expect, we can define semigroup homomorphisms which behave appropriately with respect to the inverse semigroup structure. These will be used to define inverse semigroup actions, which are essential in the remainder of this thesis.

Let $\mathcal{S}, \mathcal{W}$ be semigroups. Then $\theta : \mathcal{S} \to \mathcal{W}$ is a *semigroup homomorphism* if, for all $x, y \in \mathcal{S}$, one has

$$\theta(xy) = \theta(x)\theta(y).$$

Lemma 3.1.6. *Let $\mathcal{S}, \mathcal{W}$ be inverse semigroups and let $\theta : \mathcal{S} \to \mathcal{W}$ be a semigroup homomorphism.*

(i) If $s \in \mathcal{S}$, then $\theta(s^*) = \theta(s)^*$.

(ii) For all $e \in E(\mathcal{S})$, one has $\theta(e) \in E(\mathcal{W})$.

(iii) $\mathrm{ran}(\theta) \leq \mathcal{W}$.

Proof. *(i)* Let $x \in \mathcal{S}$. We have

$$\theta(x)\theta(x^*)\theta(x) = \theta(xx^*x) = \theta(x),$$

and so $\theta(x^*)$ is the inverse of $\theta(x)$.

(ii) Suppose $e \in E(\mathcal{S})$, and without loss of generality, let $e = s^*s$ for some $s \in \mathcal{S}$. Then, using part (i), we have

$$\theta(e) = \theta(s^*s) = \theta(s^*)\theta(s) = \theta(s)^*\theta(s),$$

and thus $\theta(e) \in E(\mathcal{W})$.

(iii) We first check that $\mathrm{ran}(\theta)$ is closed under multiplication. Let $\theta(s), \theta(t) \in \mathrm{ran}(\theta)$ for some $s, t \in \mathcal{S}$. Then $\theta(s)\theta(t) = \theta(st) \in \mathrm{ran}(\theta)$. Similarly, if $\theta(s) \in \mathrm{ran}(\theta)$, then $\theta(s)^* = \theta(s^*) \in \mathrm{ran}(\theta)$, and so $\mathrm{ran}(\theta)$ is closed under inversion. □

3.1.3 Boolean Inverse Semigroups

We refer the reader to [BBS84, Chapter IV §1] for the basics on Boolean algebras.

A *Boolean algebra* is a 5-tuple $(B, \wedge, \vee, ', 0, 1)$ whereby $\wedge, \vee$ are binary operations in B called meet and join, respectively, $'$ is a unary operation on B

called complementation, and $0, 1$ are distinguished elements such that the following hold.

(i) $(B, \vee, \wedge)$ is a distributive lattice, in the sense that the meet and join operators distribute over one another.

(ii) For all $x \in B$, one has $x \wedge 0 = 0$ and $x \vee 1 = 1$.

(iii) For all $x \in B$, $x \wedge x' = 0$ and $x \vee x' = 1$.

A *generalized Boolean algebra* is a Boolean algebra that may not contain a 1. It follows from B being complemented and distributive that it is relatively complemented - in particular, if $s, t \in B$ are such that $s \leq t$, then there exists $x \in B$ such that $x \wedge s = 0$ and $x \vee s = t$. In this case, we write $x = t \setminus s$. If E is a lattice, then for any $e, f \in E$, the join of e and f is their least upper bound, or supremum, written $e \vee f$. Conversely, the meet of e and f is their greatest lower bound, or infimum, written $e \wedge f$. If $\mathcal{S}$ is an inverse semigroup, recall that we have $e \wedge f = ef$ for all idempotents e, f.

The following details on Boolean inverse semigroups can be found in [Ste23, Section 2.1] and [Weh17, Section 3]. Let $\mathcal{S}$ be an inverse semigroup with 0. We say that $s, t \in \mathcal{S}$ are *compatible* and write $s \sim_c t$ if st^* and s^*t are both idempotents. On the other hand, if $st^* = s^*t = 0$, then we say s and t are *orthogonal* and write $s \perp t$. Notice that since 0 is an idempotent, orthogonality implies compatibility.

We say that an inverse semigroup $\mathcal{S}$ with 0 is a *Boolean inverse semigroup* if $E(\mathcal{S})$ is a generalized Boolean algebra, and all compatible elements $s \sim_c t$ admit a join. If $\mathcal{S}$ is a Boolean inverse semigroup, then in particular, this means that $E(\mathcal{S})$ is distributive, and idempotents admit relative complements.

We refer to a section of a result of Wehrung below.

Remark 3.1.7. Note that for arbitrary compatible elements $s, t \in \mathcal{S}$, the meet $s \wedge t$ doesn't necessarily coincide with the product st, as it does with idempotents. For a characterization of the meet of arbitrary inverse semigroup elements, see [Weh17, Equation 3.1.3]. In the following proposition, we denote the semigroup product by "$\cdot$" for clarity.

Proposition 3.1.8. [Weh17, Proposition 3.1.9] The following statements hold for any distributive inverse semigroup $\mathcal{S}$ with 0.

(1) For any nonempty finite compatible subset $\{b_1, \ldots, b_n\}$ of $\mathcal{S}$, the join $\bigvee_{i=1}^n b_i$ exists and the following statements hold.

   (i) For every $a \in \mathcal{S}$, $a \cdot (\bigvee_{i=1}^n b_i) = \bigvee_{i=1}^n (a \cdot b_i)$ and $(\bigvee_{i=1}^n b_i) \cdot a = \bigvee_{i=1}^n (b_i \cdot a)$.

   (ii) For every $a \in \mathcal{S}$, $a \wedge \bigvee_{i=1}^n b_i$ exists if and only if each $a \wedge b_i$ exists, and then $a \wedge \bigvee_{i=1}^n b_i = \bigvee_{i=1}^n (a \wedge b_i)$.

3.1.4 Examples

Example 3.1.9. We continue investigating symmetric inverse semigroups, as introduced in Example 3.1.5. Let X be a set. One can see that if $s, t \in \mathcal{I}(X)$ are partial bijections, then $s \sim_c t$ if and only if

$$s|_{\mathrm{dom}(s)\cap\mathrm{dom}(t)} = t|_{\mathrm{dom}(s)\cap\mathrm{dom}(t)}.$$

That is, if they agree on their common domain. Furthermore, $s \perp t$ if and only if $\mathrm{dom}(s) \cap \mathrm{dom}(t) = \emptyset$ and $\mathrm{ran}(s) \cap \mathrm{ran}(t) = \emptyset$.

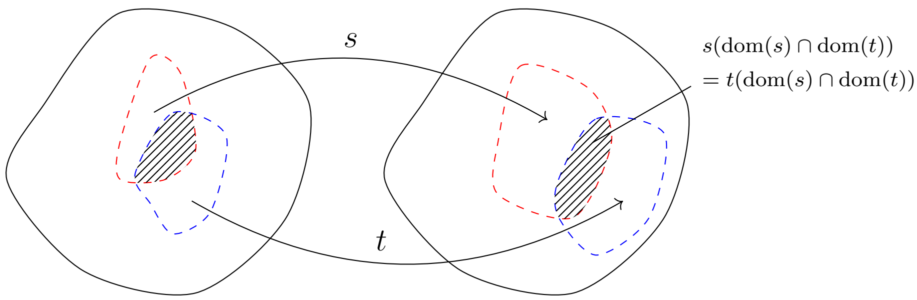


Figure 3.2: A visual representation of two compatible partial bijections.

In general, any finite symmetric inverse semigroup is Boolean [Weh18, Section 2].

3.2 Inverse Semigroup Actions

The following definitions can be found in [Exe08, Section 4].

Definition 3.2.1. Let $\mathcal{S}$ be a semigroup and X a locally compact Hausdorff space. An *action* of $\mathcal{S}$ on X is a semigroup homomorphism

$$\theta : \mathcal{S} \to \mathcal{I}(X)$$

satisfying the following.

(A1) For every $s \in \mathcal{S}$, $\theta(s)$ is continuous and $\mathrm{dom}(\theta(s))$ is open in X.

(A2) Whenever $x \in X$, there exists $s \in \mathcal{S}$ such that $x \in \mathrm{dom}(\theta(s))$.

For each $s \in \mathcal{S}$, we let $\theta_s : X \to X$ denote the partial bijection $\theta(s)$. Hence, the action can also be thought of as a function

$$\bigcup_{s\in\mathcal{S}}(\{s\} \times \mathrm{dom}(\theta(s))) \to \bigcup_{s\in\mathcal{S}} \mathrm{ran}(\theta(s)) = X.$$

In this way, if $x \in \mathrm{dom}(\theta_s)$ then

$$(s, x) \mapsto \theta_s(x).$$

For an idempotent $e \in E(\mathcal{S})$, we denote by D_e the domain and range of θ_e. Notice also that the map θ_s for some $s \in \mathcal{S}$ has domain D_{s^*s} and range D_{ss^*}. This is because $\mathrm{dom}(\theta_s) = \mathrm{dom}(\theta_{s^*}\theta_s)$, and similarly, $\mathrm{ran}(\theta_s) = \mathrm{ran}(\theta_s\theta_{s^*})$.

The following lemmas follow from straightforward calculations, the details of which can be found in [Exe08, Section 4].

Lemma 3.2.2. *Let $s \in \mathcal{S}$ and $e \in E(\mathcal{S})$.*

(i) $\theta_s\theta_{s^*}\theta_s = \theta_s$.

(ii) $\theta_{s^*}\theta_s\theta_{s^*} = \theta_{s^*}$.

(iii) $\theta_{s^*} = \theta_s^{-1}$.

(iv) $\operatorname{ran}(\theta_s) = \operatorname{dom}(\theta_{s^*})$.

(v) $\theta_e = \operatorname{id}_{\operatorname{dom}(\theta_e)}$.

(vi) $\operatorname{dom}(\theta_s) = \operatorname{dom}(\theta_{s^*s})$ *and* $\operatorname{ran}(\theta_s) = \operatorname{ran}(\theta_{ss^*})$.

3.2.1 Groupoids of Germs

The following groupoid of germs construction follows that of [Exe08] and [MR10].

Consider an arbitrary locally compact Hausdorff space X under an action by $\mathcal{S}$. We define a set of ordered pairs

$$\Omega = \{(s, x) \in \mathcal{S} \times X : x \in D_{s^*s}\}.$$

That is, Ω is the set of pairs consisting of elements $s \in \mathcal{S}$ along with points $x \in D_{s^*s}$. We define an equivalence relation on Ω by setting $(s, x) \sim (t, y)$ if and only if $x = y$, and there exists some idempotent $e \in E(\mathcal{S})$ such that $x \in D_e$ and $se = te$. Loosely speaking, two pairs (s, x) and (t, x) are considered equivalent if s and t act identically on some open neighbourhood of x corresponding to the domain of an idempotent.

The quotient $\Omega/\sim$ is called the *groupoid of germs* of the action of $\mathcal{S}$ on X, and is denoted $\mathcal{S} \ltimes X$. The composable pairs of $\mathcal{S} \ltimes X$ are

$$(\mathcal{S} \ltimes X)^{(2)} = \{([s, x], [t, y]) \in (\mathcal{S} \ltimes X) \times (\mathcal{S} \ltimes X) : x = \theta_t(y)\},$$

and the inverse operation is defined by $[s, x]^{-1} = [s^*, \theta_s(x)]$. Composition is defined as $[s, x][t, y] = [st, y]$ for any $([s, x], [t, y]) \in (\mathcal{S} \ltimes X)^{(2)}$, and it follows that the source and range maps are given by

$$d([s, x]) = [s^*s, x] \quad \text{and} \quad r([s, x]) = [ss^*, \theta_s(x)].$$

The following proposition is stated without proof in [Exe08, Proposition 4.11]. We provide details below.

Proposition 3.2.3. The construction $\mathcal{S} \ltimes X$ is a groupoid with the operations as defined as above, and $(\mathcal{S} \ltimes X)^{(0)} \cong X$ under the correspondence

$$[e, x] \in (S \ltimes X)^{(0)} \mapsto x \in X,$$

where e is an idempotent such that $x \in D_e$.

Proof. We check that $\mathcal{S} \ltimes X$ satisfies the groupoid formulation stated in Definition 2.1.1.

First, we check that for any $\gamma \in \mathcal{S} \ltimes X$, we have $(\gamma^{-1})^{-1} = \gamma$. Let $[s, x] \in \mathcal{S} \ltimes X$. Then,

$$\begin{aligned} ([s, x]^{-1})^{-1} &= ([s^*, \theta_s(x)])^{-1} \\ &= [(s^*)^*, \theta_{s^*}(\theta_s(x))] \\ &= [s, \theta_s^{-1}(\theta_s(x))] \\ &= [s, x]. \end{aligned}$$

Next, we check that if $([s, x], [t, y])$ and $([t, y], [u, z])$ are in $(\mathcal{S} \ltimes X)^{(2)}$, then so are $([s, x][t, y], [u, z])$ and $([s, x], [t, y][u, z])$, and associativity holds. The first claim is straightforward to check, as

$$([s, x][t, y], [u, z]) = ([st, y], [u, z]),$$

and $([t, y], [u, z]) \in (\mathcal{S} \ltimes X)^{(2)}$ implies $\theta_u(z) = y$, and so $([st, y], [u, z]) \in (\mathcal{S} \ltimes X)^{(2)}$. Similarly,

$$([s, x], [t, y][u, z]) = ([s, x], [tu, z]),$$

and $\theta_{tu}(z) = \theta_t(\theta_u(z)) = \theta_t(y) = x$ as needed. We can then see that

$$\begin{aligned} ([s, x][t, y])[u, z] &= [st, y][u, z] \\ &= [stu, z] \\ &= [s, x][tu, z] \\ &= [s, x]([t, y][u, z]). \end{aligned}$$

For the last groupoid condition, we begin by noticing $[s,x][s,x]^{-1} \in (\mathcal{S} \ltimes X)^{(2)}$ since $\theta_s^{-1}(\theta_s(x)) = x$. Furthermore, $(st)^*st$ is an idempotent satisfying the following.

$$\begin{aligned}(s^*st)((st)^*st) &= s^*stt^*s^*st \\ &= tt^*s^*ss^*st \\ &= tt^*s^*st \\ &= t((st)^*st).\end{aligned}$$

That is, if $x \in D_{(st)^*st}$, then $[s^*st, x] = [t, x]$. It follows that

$$\begin{aligned}[s,x]^{-1}([s,x][t,y]) &= [s^*, \theta_s(x)][st, y] \\ &= [s^*st, y] \\ &= [t, y].\end{aligned}$$

Similarly, the following equivalence of germs is witnessed by the idempotent tt^* - notice that $stt^* = stt^*tt^*$, and so we have

$$\begin{aligned}([s,x][t,y])[t,y]^{-1} &= [st, y][t^*, \theta_t(y)] \\ &= [stt^*, \theta_t(y)] \\ &= [s, x].\end{aligned}$$

Hence $\mathcal{S} \ltimes X$ is a groupoid.

It remains to show that $(\mathcal{S} \ltimes X)^{(0)} \cong X$ under the given correspondence. Let $\phi : (\mathcal{S} \ltimes X)^{(0)} \to X$ be the map given by $[e,x] \mapsto x$. We claim that ϕ is a bijection. If $x \in X$, then there exists $s \in \mathcal{S}$ such that $x \in D_{s^*s}$. Thus, $[s^*s, x] \mapsto x$, and so ϕ is bijective. Now, suppose $[e,x], [f,x] \in \mathcal{S} \ltimes X$. Then $x \in D_e \cap D_f = D_{ef}$, and $eef = fef = ef$, implying $[e,x] = [f,x]$. Hence, ϕ is injective. □

Henceforth, for any $[s,x] \in \mathcal{S} \ltimes X$, we identify $d([s,x]) = [s^*s, x]$ with $x \in X$, and $r([s,x]) = [ss^*, \theta_s(x)]$ with $\theta_s(x) \in X$.

3.2.2 Groupoids of Germs are Étale

We now turn $\mathcal{S} \ltimes X$ into a topological groupoid by defining a basis on it. For any $s \in \mathcal{S}$ and open $U \subseteq D_{s^*s}$ (with respect to the topology on X) we define

$$\Theta(s, U) = \{[s, x] \in \mathcal{S} \ltimes X : x \in U\}.$$

We attribute details of the following to [Exe08, Proposition 4.13].

Proposition 3.2.4. The collection of sets $\Theta(s, U)$ for $s \in \mathcal{S}$ and $U \subseteq_\circ D_{s^*s}$ is a basis for the topology on $\mathcal{S} \ltimes X$.

Proof. If $[s, x] \in \mathcal{S} \ltimes X$, then $x \in D_{s^*s}$ which is an open set. Thus, $[s, x] \in \Theta(s, D_{s^*s})$.

It remains to show that if $s, t \in \mathcal{S}$, with U, V open subsets of D_{s^*s} and D_{t^*t} respectively, and $[r, z] \in \Theta(s, U) \cap \Theta(t, V)$, then there exists an idempotent e and open set $W \subseteq D_{(re)^*re}$ such that

$$[r, z] \in \Theta(re, W) \subseteq \Theta(s, U) \cap \Theta(t, V).$$

Since $[r, z] \in \Theta(s, U) \cap \Theta(t, V)$, we have $[r, z] = [s, x] = [t, y]$ for some $x \in U$ and $y \in V$. This implies $z = x = y$, and furthermore there exist $e, f \in E(\mathcal{S})$ such that $z \in D_e \cap D_f$, $re = se$ and $rf = tf$. Since idempotents commute, we can say without loss of generality that $re = se = te$. Let $W = U \cap V \cap D_{(re)^*re}$. Then

$$z \in D_{r^*r} \cap D_e = D_{r^*re} = D_{(re)^*re},$$

which follows from

$$(re)^*re = e^*r^*re = r^*re^*e = r^*re.$$

Hence, we have $z \in W$ and so $z \in \Theta(re, W)$. It remains to show that $\Theta(re, W) \subseteq \Theta(s, U) \cap \Theta(t, V)$. Let $[re, x] \in \Theta(re, W)$ be arbitrary. Then

$$[re, x] = [se, x] = [s, x],$$

and similarly,

$$[re, x] = [te, x] = [t, x],$$

and so $[re, x] \in \Theta(s, U) \cap \Theta(t, V)$ as desired. □

This shows that the collection of sets of the form $\Theta(s, U)$ form a basis for $\mathcal{S} \ltimes X$. One can verify that with respect to this topology, the multiplication and inverse maps are continuous, and hence $\mathcal{S} \ltimes X$ is a topological groupoid.

Henceforth, if $s \in \mathcal{S}$, we denote Θ_s the set

$$\Theta_s := \Theta(s, D_{s^*s}).$$

Proposition 3.2.5. [Exe08, Proposition 4.14] With the topology generated by the basis above, $\mathcal{S} \ltimes X$ is a topological groupoid.

Since we are interested in the class of étale groupoids, we would like for $\mathcal{S} \ltimes X$ to also be an étale groupoid. First, we construct a local correspondence between open subsets of X and basic open sets of $\mathcal{S} \ltimes X$.

Proposition 3.2.6. For some $s \in \mathcal{S}$ and open set $U \subseteq D_{s^*s}$, the map

$$\phi : U \to \Theta(s, U), \quad x \mapsto [s, x]$$

is a homeomorphism.

Proof. It follows directly from the construction of the equivalence classes $[s, x]$ that ϕ is bijective. Note that ϕ has a well-defined inverse, such that if $s \in \mathcal{S}$ and $[s, x] \in \mathcal{S} \ltimes X$, then

$$\phi^{-1}([s, x]) = x,$$

which is precisely the source mapping. Let $V \subseteq U$ be open. Then $\phi(V) = \{[s, x] : x \in V\} = \Theta(s, V)$ which is a basis element, and so is open. Therefore, ϕ is an open map - that is, ϕ^{-1} is continuous). We now show that ϕ is continuous. Let $\phi(x) = [s, x]$ be in some open $W \subseteq \Theta(s, U)$. Then, there is some basis element $\Theta(t, V)$ such that

$$[s, x] \in \Theta(t, V) \subseteq W.$$

Hence, we have that $x \in V \subseteq D_{t^*t}$, and there exists an idempotent $e \in E(\mathcal{S})$ such that $se = te$. For any $y \in D_e \cap V$,

$$\phi(y) = [s, y] = [t, y] \in \Theta(t, V) \subseteq W,$$

indicating $\phi(D_e \cap V) \subseteq W$. But $x \in D_e \cap V$, and so ϕ is continuous. □

The following is a special case of the above result.

Corollary 3.2.7. The identification of $(\mathcal{S} \ltimes X)^{(0)}$ with X given by

$$x \mapsto [e, x],$$

where e is any idempotent such that $x \in D_e$, is a homeomorphism.

One can show that our basic open sets are in fact bisections [Exe08, Proposition 4.18].

Proposition 3.2.8. For every $s \in \mathcal{S}$ and open $U \subseteq D_{s^*s}$, the set $\Theta(s, U)$ is an open bisection.

Proof. The source map equates with ϕ^{-1} when restricted to $\Theta(s, U)$, and so is injective. Regarding the range map, we see that for $x \in \Theta(s, U)$, we have

$$r([s, x]) = \theta_s(x) = (\theta_s \circ d)(x),$$

and so is injective, being the composition of two injective maps. □

We can now show the following result, which is stated in [Exe08], but we fill in the details.

Proposition 3.2.9. The groupoid of germs $\mathcal{S} \ltimes X$ is an étale groupoid.

Proof. Since X is Hausdorff and locally compact, $(\mathcal{S} \ltimes X)^{(0)}$ is also Hausdorff and locally compact, being identified with X via a homeomorphism. If $[s, x] \in (\mathcal{S} \ltimes X)$, then there exists some open bisection $\Theta(s, U)$ such that $[s, x] \in \Theta(s, U)$. Then, the range and source maps restricted to $\Theta(s, U)$ are homeomorphisms. Hence, $\mathcal{S} \ltimes X$ is étale.

□

3.2.3 Relationships Between $\mathcal{S}$ and $\mathcal{S} \ltimes \mathcal{G}^{(0)}$

There is a clear relationship between the fixed points of the action of an inverse semigroup on a locally compact Hausdorff space, and the isotropy subgroupoid of the arising groupoid of germs. In the literature regarding these actions, such as in [KM21, Section 2.4], [CB20, Section 7] and [EP16, Section 4], the action of an inverse semigroup $\mathcal{S}$ on a space X is called topologically free, effective, or topologically principal, if the groupoid of germs [1] is topologically free, effective, or topologically principal, respectively. In this section, we determine under what conditions an action satisfies these conditions.

Recall that an inverse semigroup $\mathcal{S}$ is E^**-unitary* if, whenever $e \in E(\mathcal{S})$ and $s \in \mathcal{S}$ such that $0 < e \leq s$, then $s \in E(\mathcal{S})$ [Law98, p. 19].

Let $\mathcal{S}$ be an inverse semigroup, X a locally compact Hausdorff space, and θ an action of $\mathcal{S}$ on X with groupoid of germs $\mathcal{S} \ltimes X$.

Proposition 3.2.10. If $\mathcal{S}$ is E^*-unitary, then $\mathcal{S} \ltimes X$ is Hausdorff.

Proof. Toward a contradiction, suppose that $\mathcal{S} \ltimes X$ is not Hausdorff. By Proposition 2.2.1, this is equivalent to $(\mathcal{S} \ltimes X)^{(0)}$ not being closed , but note from Corollary 3.2.9 that it is still open due to $\mathcal{S} \ltimes X$ being étale. Since $(\mathcal{S} \ltimes X)^{(0)}$ is not closed, we have $\partial(\mathcal{S} \ltimes X)^{(0)} \neq \emptyset$, and since $(\mathcal{S} \ltimes X)^{(0)}$ is open, $\partial(\mathcal{S} \ltimes X)^{(0)} \cap (\mathcal{S} \ltimes X)^{(0)} = \emptyset$ [Sim63, Section 17]. If $[s, x] \in \partial(\mathcal{S} \ltimes X)^{(0)}$,

[1] This groupoid is called the *transformation groupoid* in [KM21].

then $[s,x] \in \mathcal{S} \ltimes X \setminus (\mathcal{S} \ltimes X)^{(0)}$. Thus, s is a non-idempotent of $\mathcal{S}$. To see this, using Corollary 3.2.7, notice that there exists no $e \in E(\mathcal{S})$ such that $x \in D_e$ and $se = e$. Consdering $s^*s \in E(\mathcal{S})$, we have $x \in D_{s^*s}$, and so we must have $ss^*s = s \neq s^*s$.

Let $\Theta(s, A)$ be an open neighbourhood of $[s, x]$, where $A \subseteq D_{s^*s}$ is open and $x \in A$. By definition of the boundary, $\Theta(s, A)$ contains some unit $[e, z] \in (\mathcal{S} \ltimes X)^{(0)}$ where e is an idempotent and $z \in D_e$. But $[e, z] \in \Theta(s, A)$, and so $[e, z] = [s, w]$ for some $w \in A$.

By definition of germ equivalence, we have that $z = w$ and there exists some non-zero idempotent e' such that $ee' = se'$. Then, $ee'e = se'e$ which implies $ee' = see'$, and so $s \geq ee'$. Since $\mathcal{S}$ is E^*-unitary, this implies s is an idempotent, which contradicts the fact that $[s, x]$ lies outside $(\mathcal{S} \ltimes X)^{(0)}$. Hence, $\mathcal{S} \ltimes X$ is Hausdorff. □

Recall that if $\mathcal{S} \ltimes X$ is topologically principal then it is topologically free [CEP+19]. It is known that in the case where $\mathcal{S} \ltimes X$ is second-countable, effective implies topologically principal, and upon the addition of the Hausdorff assumption, all three conditions become equivalent.

If $\theta_s(x) = x$, then we say s is a *stabilizer* of x, and we say s is a *non-trivial stabilizer* of x if $s \notin E(\mathcal{S})$. Following [CB20, Definition 7.2], we say that x is a *fixed point* of $s \in \mathcal{S}$ if $\theta_s(x) = x$, and we say that x is a *trivially fixed point* of s if there exists an idempotent $e \leq s$ such that $x \in D_e$. In this case, one has

$$\theta_s(x) = \theta_s(\theta_e(x)) = \theta_{se}(x) = \theta_e(x) = x,$$

and so x is indeed a fixed point of s. We denote by $\mathrm{Fix}(s)$ and $\mathrm{TFix}(s)$ the set of fixed points and the set of trivially fixed points of s, respectively. We note that $\mathrm{Fix}(s)$ corresponds to elements in $\mathrm{Iso}(\mathcal{G})$, whereas elements of $\mathrm{TFix}(s)$ correspond to units in $\mathcal{G}^{(0)}$. This motivates us to consider the non-trivially fixed points, $\mathrm{Fix}(s) \setminus \mathrm{TFix}(s)$. Hence, we define a family of sets as follows.

$$\mathcal{F}_s := \{x \in D_{s^*s} : x \in \mathrm{Fix}(s) \setminus \mathrm{TFix}(s)\}, \quad s \in \mathcal{S}.$$

That is, $\mathcal{F}_s$ is the set of "non-trivially" fixed points of s. Furthermore, we define

$$\mathcal{F} := \bigcup_{s \in \mathcal{S}} \mathcal{F}_s.$$

Notice that $\mathcal{F}$ is exactly the collection of units in the groupoid of germs with non-trivial isotropy.

The following result is adapted from [EP16, Theorem 4.7].

Proposition 3.2.11. The groupoid $\mathcal{S} \ltimes X$ is effective if and only if for every non-idempotent $s \in \mathcal{S}$, the interior of $\mathrm{Fix}(s)$ coincides with $\mathrm{TFix}(s)$.

Proof. We assume $\mathcal{S} \ltimes X$ is effective. Since $\mathrm{TFix}(s)$ is open in $\mathrm{Fix}(s)$ (see [EP16, Equation 4.2]), we automatically have $\mathrm{TFix}(s) \subseteq \mathrm{Int}(\mathrm{Fix}(s))$, so it suffices to show the reverse inclusion holds. Let $x \in \mathrm{Int}(\mathrm{Fix}(s))$, and so there exists an open set $U \ni x$ such that $U \subseteq \mathrm{Fix}(s)$. That is, for every $z \in U$, one has $\theta_s(z) = z$. This corresponds to the open set $\Theta(s, U) \subseteq \mathrm{Iso}(\mathcal{S} \ltimes X)$ in $\mathcal{S} \ltimes X$, and so $\Theta(s, U) \subseteq \mathrm{Int}(\mathrm{Iso}(\mathcal{S} \ltimes X))$. Since $\mathcal{S} \ltimes X$ is effective, we have $\Theta(s, U) \subseteq (\mathcal{S} \ltimes X)^{(0)}$, and in particular $[s, x] \in (\mathcal{S} \ltimes X)^{(0)}$, which implies $[s, x] = [e, x]$ for some idempotent e. Hence, there is some idempotent e' such that $x \in D_{e'}$ and $se' = ee'$, from which we have

$$see' = se'e = ee'e = ee',$$

and so $ee' \leq s$. Furthermore, $x \in D_e \cap D_{e'} = D_{ee'}$, and so $x \in \mathrm{TFix}(s)$ as desired. Thus, we have shown that $\mathrm{TFix}(s) = \mathrm{Int}(\mathrm{Fix}(s))$.

Conversely, suppose that $\mathrm{Int}(\mathrm{Fix}(s)) = \mathrm{TFix}(s)$. It suffices to show that $\mathrm{Int}(\mathrm{Iso}(\mathcal{S} \ltimes X)) \subseteq (\mathcal{S} \ltimes X)^{(0)}$, so let $[s, x] \in \mathrm{Int}(\mathrm{Iso}(\mathcal{S} \ltimes X))$. Then, there is some open set $\Theta(s, U) \ni [s, x]$ contained in $\mathrm{Iso}(\mathcal{S} \ltimes X)$ - that is, for all $z \in U$, one has $\theta_s(z) = z$. Therefore $U \subseteq \mathrm{Int}(\mathrm{Fix}(s)) = \mathrm{TFix}(s)$, and so

$[s,x] \in \mathrm{TFix}(s)$. Then there exists an idempotent $e \leq s$ such that $x \in D_e$. Hence, $e = se$ which implies $ee = se$, and so by germ equivalence we have $[s,x] = [e,x]$. Then $[s,x] \in (\mathcal{S} \ltimes X)^{(0)}$ as desired, and $\mathcal{S} \ltimes X$ is effective. $\square$

Proposition 3.2.12. The groupoid $\mathcal{S} \ltimes X$ is topologically free if and only if, for each $s \in \mathcal{S}$, the set $\mathcal{F}_s$ has empty interior.

Proof. Suppose $\mathcal{S} \ltimes X$ is not topologically free. Then, there exists some $[s,x] \in$ Int(Iso$(\mathcal{S} \ltimes X) \setminus (\mathcal{S} \ltimes X)^{(0)}$). Hence, there is some open set $\Theta(s,U)$ containing $[s,x]$ and contained within $\mathrm{Iso}(\mathcal{S} \ltimes X) \setminus (\mathcal{S} \ltimes X)^{(0)}$. This means that $U \subseteq \mathrm{Fix}(s)$, and since $\Theta(s,U)$ is disjoint from $(\mathcal{S} \ltimes X)^{(0)}$, if $[s,z] \in \Theta(s,U)$, then there exists no idempotent e such that $z \in D_e$ and $se = e$. That is, $z \notin \mathrm{TFix}(s)$. This implies $U \subseteq \mathrm{Fix}(s) \setminus \mathrm{TFix}(s) = \mathcal{F}_s$, and so $\mathcal{F}_s$ has non-empty interior.

Conversely, suppose there exists $s \in \mathcal{S}$ such that Int$(\mathcal{F}_s) \neq \emptyset$. Hence, there exists an open set $U \subseteq \mathcal{F}_s = \mathrm{Fix}(s) \setminus \mathrm{TFix}(s)$, meaning that U consists entirely of non-trivially fixed points of s. In particular, for every $x \in U$, we have $\theta_s(x) = x$, but there exists no idempotent $e \leq s$ with $x \in D_e$. This implies that there exists no $[e,x] \in (\mathcal{S} \ltimes X)^{(0)}$ such that $[s,x] = [e,x]$, and so $[s,x] \in \mathrm{Iso}(\mathcal{S} \ltimes X) \setminus (\mathcal{S} \ltimes X)^{(0)}$. Thus, $U \subseteq$ Int(Iso$(\mathcal{S} \ltimes X) \setminus (\mathcal{S} \ltimes X)^{(0)}$), and $\mathcal{S} \ltimes X$ is not topologically free. $\square$

Proposition 3.2.13. The set $\mathcal{F}$ has empty interior in X if and only if $\mathcal{S} \ltimes X$ is topologically principal.

Proof. We begin by assuming that $\mathcal{F}$ has empty interior in X, and let $\Theta(e,U) \subseteq (\mathcal{S} \ltimes X)^{(0)}$ be open. Toward a contradiction, suppose that $\Theta(e,U)$ consists entirely of non-trivial isotropy. That is, whenever $x \in U$, there exists $s \in \mathcal{S}$ such that $x \in D_{s^*s}$ and $x \in \mathrm{TFix}(s) \subseteq \mathrm{Fix}(s)$. But then $U \subseteq \mathcal{F}$, contradicting our assumption that $\mathcal{F}$ has empty interior.

Conversely, suppose that $\mathcal{S} \ltimes X$ is topologically principal, and let $U \subseteq \mathcal{F}$

be open. For any $x \in U$, we have some $s \in \mathcal{S}$ such that $x \in \mathcal{F}_s$ - that is, x is a non-trivially fixed point of s. This is true for any x we choose, and so U consists entirely of non-trivially fixed points, which are exactly those units with non-trivial isotropy. Then $\Theta(e, U)$ is entirely units with non-trivial isotropy, which contradicts $\mathcal{S} \ltimes X$ being topologically principal. $\square$

For any $x \in (\mathcal{S} \ltimes X)^{(0)}$, the *orbit* of x is the set of units y such that, for some $\gamma \in \mathcal{S} \ltimes X$, $d(\gamma) = x$ and $r(\gamma) = y$. We denote this set by $\mathrm{Orb}(x)$.

We introduce a property of an action θ of $\mathcal{S}$ on a space X called *local transitivity*, which is a weaker condition than transitivity. We say an action is locally transitive if, for every $x \in X$ and non-empty open $U \subset X$, there exists $s \in \mathcal{S}$ such that $\theta_s(x) \in U$. As opposed to transitivity, which requires that for every $y \in X$ there exists some $s \in \mathcal{S}$ such that $\theta_s(x) = y$, local transitivity only requires that there exists some $s \in \mathcal{S}$ that carries x arbitrarily close to y. An alternate formulation of local transitivity is for every $x, y \in X$, there is a subcollection $T \subseteq \mathcal{S}$ such that the net $(\theta_s(x))_{s \in T}$ converges to y.

Recall that a set $U \subseteq (\mathcal{S} \ltimes X)^{(0)}$ is *invariant* whenever $d(\gamma) \in U$ if and only if $r(\gamma) \in U$ for all $\gamma \in \mathcal{S} \ltimes X$.

Lemma 3.2.14. *A set $U \subseteq (\mathcal{S} \ltimes X)^{(0)}$ is invariant if and only if $(\mathcal{S} \ltimes X)^{(0)} \setminus U$ is invariant.*

Proof. If U isn't invariant, then $G_U \neq G^U$. Without loss of generality, suppose $r(G_U) \supset U$. Since inverses exist, for any $x \in r(G_U) \setminus U$ we have $r(x) \in U$, implying $(\mathcal{S} \ltimes X)^{(0)} \setminus U$ isn't invariant. The converse follows similarly from taking complements. $\square$

Lemma 3.2.15. *A groupoid $\mathcal{S} \ltimes X$ is minimal if and only if, for every $x \in (\mathcal{S} \ltimes X)^{(0)}$, $\mathrm{Orb}(x)$ is dense in $(\mathcal{S} \ltimes X)^{(0)}$.*

Proof. First, suppose that for any $x \in (\mathcal{S} \ltimes X)^{(0)}$, $\mathrm{Orb}(x)$ is dense in the unit space. Pick some open $U \subseteq (\mathcal{S} \ltimes X)^{(0)}$. If $x \in U$, then pick a different $V \subseteq (\mathcal{S} \ltimes X)^{(0)} \setminus U$. So, we have $x \notin U$. Since $\mathrm{Orb}(x)$ is dense, there is some $s \in \mathcal{S}$ and $[s, x] \in \mathcal{S} \ltimes X$ such that $r([s, x]) \in U$. Hence, U cannot be invariant, and so $\mathcal{S} \ltimes X$ is minimal.

Conversely, suppose $\mathcal{S} \ltimes X$ is minimal, and pick some $x \in (\mathcal{S} \ltimes X)^{(0)}$. We aim to show that $\mathrm{Orb}(x)$ is dense in $(\mathcal{S} \ltimes X)^{(0)}$. Pick some open $U \subseteq (\mathcal{S} \ltimes X)^{(0)}$ not containing x. We claim that the sets $r(\mathcal{S} \ltimes X)$ and U have non-empty intersection. Since we can compose arrows, there exists no $\gamma \in \mathcal{S} \ltimes X$ such that $d(\gamma) \in r(\mathcal{S} \ltimes X)$ and $r(\gamma) \in (\mathcal{S} \ltimes X)^{(0)} \setminus r(\mathcal{S} \ltimes X)$ and so $r(\mathcal{S} \ltimes X)$ is in fact invariant. This implies the minimality of $\mathcal{S} \ltimes X$, a contradiction, meaning $r(\mathcal{S} \ltimes X) \cap U \neq \emptyset$ as desired. □

Proposition 3.2.16. The action θ of $\mathcal{S}$ on X is locally transitive on X if and only if $\mathcal{S} \ltimes X$ is minimal.

Proof. Suppose $\mathcal{S} \ltimes X$ is not minimal. Then, there is some $x \in (\mathcal{S} \ltimes X)^{(0)}$ and open set $U \subseteq (\mathcal{S} \ltimes X)^{(0)}$ such that

$$\{r([s, x]) : x \in D_{s^*s}\} \cap U = \emptyset.$$

In other words, for every $u \in U$, there is no arrow from x to u. Hence, U is a set disjoint from $\mathrm{Orb}(x)$, and so $\mathcal{S}$ isn't locally transitive.

Now, suppose $\mathcal{S}$ is not locally transitive. Then, there is some $x \in X$ and open $U \subseteq X$ such that $\mathrm{Orb}(x)$ with respect to $\mathcal{S}$ is disjoint from U. Hence, $\mathrm{Orb}(x)$ is not dense in $(\mathcal{S} \ltimes X)^{(0)}$, and so $\mathcal{S} \ltimes X$ is not minimal. □

While the equivalence between locally transitive and minimal is straightforward, it is useful as it provides a characterization of minimality in groupoids with respect to the acting inverse semigroup, without needing to mention the groupoid of germs.

3.3 The Natural Action of $\mathrm{Bis}(\mathcal{G})$ on $\mathcal{G}^{(0)}$

The motivation for this section arose from wishing to provide full details of the proof of [BM23, Proposition 4.12], which was also investigated by Exel in [Exe08, Proposition 5.4]. This result shows that every étale groupoid possesses an *intrinsic* inverse semigroup - its inverse semigroup of open bisections - and shows that information about this inverse semigroup can be discarded while retaining the ability to recover the original groupoid as the groupoid of germs of a natural action on the unit space. In particular, the original groupoid can be recovered as long as the sub-semigroup satisfies a basis-like condition called *bisection wide*[2]. This naturally leads to an investigation of the more general case, where an inverse semigroup acts on a locally compact Hausdorff space, giving rise to an étale groupoid of germs, whereby it becomes natural to study the relationships between properties of the inverse semigroup and the groupoid it produces.

Let $\mathcal{G}$ be an étale groupoid. We denote by $\mathrm{Bis}(\mathcal{G})$ the collection of open bisections of $\mathcal{G}$. This collection forms an inverse semigroup, where multiplication and inversion are given point-wise - this fact is well-established in the literature (see, for example, [Pat99, Proposition 2.2.4], [BM23, p. 23], [CB20, Section 2.4]). There exists a natural action of $\mathrm{Bis}(\mathcal{G})$ on $\mathcal{G}^{(0)}$ such that the groupoid of germs of this action is isomorphic to $\mathcal{G}$ (see [BM23, p. 23]). In fact, one can take a *sufficiently large* sub-semigroup $S \subseteq \mathrm{Bis}(\mathcal{G})$, and achieve the same outcome.

Given an open bisection $U \in \mathrm{Bis}(\mathcal{G})$, since the source and range maps are both open maps, the sets $d(U)$ and $r(U)$ are open in $\mathcal{G}^{(0)}$. Furthermore, both r_U and d_U are homeomorphisms on their domain. We denote by d_U and r_U the restrictions of the domain and range maps to the set U, respectively.

W define an action of $\mathrm{Bis}(\mathcal{G})$ on $\mathcal{G}^{(0)}$ whereby if $S \in \mathrm{Bis}(\mathcal{G})$ and $x \in \mathcal{G}^{(0)}$,

[2] The authors of [Exe08], along with those of other literature discussing this property, call it simply *wide*, but we append "bisection" to distinguish it from our new condition.

then

$$\theta_S(x) = r(d_S^{-1}(x)).$$

Proposition 3.3.1. Let $S \in \mathrm{Bis}(\mathcal{G})$. Then the correspondence $x \mapsto \theta_S(x)$ as described above is an action on $\mathcal{G}^{(0)}$.

Proof. We first check that $\theta_S\theta_T = \theta_{ST}$ for all $S, T \in \mathcal{S}$. For $x \in D_{(ST)^*(ST)}$, we have

$$\theta_S(\theta_T(x)) = \theta_S(r(d_T^{-1}(x))) = r(d_S^{-1}(r(d_T^{-1}(x)))).$$

Let $\gamma_T := d_T^{-1}(x)$ and $\gamma_S := d_S^{-1}(r(d_T^{-1}(x)))$. Note that $\gamma_T \in T$ and $\gamma_S \in S$. So, we have $r(\gamma_S) = d(\gamma_T)$, and so via composition, there exists an arrow $\gamma_{ST} \in ST$ such that $s(\gamma_{ST}) = x$ and $r(\gamma) = \theta_S(\theta_T(x))$. Hence, $\theta_{ST} = \theta_S\theta_T$.

It remains to check that the domain of the action covers the unit space. Since $\mathcal{G}^{(0)}$ itself is an open bisection, one has that whenever $x \in \mathcal{G}^{(0)}$, then $x \in D_{\mathcal{G}^{(0)}}$. □

The following topological basis-like condition establishes precisely what it means for a subsemigroup of $\mathrm{Bis}(\mathcal{G})$ to be large enough, and capture enough information, to reconstruct $\mathcal{G}$. This definition follows that of [BM23, Definition 4.11], however we use the term "bisection wide" to distinguish it from a later definition.

Definition 3.3.2. Let $\mathcal{S} \subseteq \mathrm{Bis}(\mathcal{G})$ be an inverse subsemigroup. Then $\mathcal{S}$ is *bisection wide* if it satisfies the following criteria.

(BW1) For every $\gamma \in \mathcal{G}$, there exists $B \in S$ such that $\gamma \in B$.

(BW2) For any $\gamma \in \mathcal{G}$ and $B, B' \in \mathcal{S}$, if $\gamma \in B \cap B'$ then there exists a $B'' \in \mathcal{S}$ such that $\gamma \in B'' \subseteq B \cap B'$.

We have already seen that $\theta : \mathrm{Bis}(\mathcal{G}) \times \mathcal{G}^{(0)} \to \mathcal{G}^{(0)}$ is a well-defined action on $\mathcal{G}^{(0)}$ - we denote the groupoid of germs of this action by $\mathcal{S} \ltimes \mathcal{G}^{(0)}$, following [BM23]. The following result is [Exe08, Proposition 5.4] (see also [BM23, Proposition 4.12]).

Theorem 3.3.3. Let $\mathcal{G}$ be a locally compact, étale groupoid, and let $\mathcal{S} \subseteq \mathrm{Bis}(\mathcal{G})$ be an inverse sub-semigroup. Then, the following are equivalent.

(i) $\mathcal{S}$ is bisection wide.

(ii) $\mathcal{S} \ltimes \mathcal{G}^{(0)} \cong \mathcal{G}$.

Proof. We first assume that $\mathcal{S}$ is bisection wide. For any germ $[U, x]$, since U is a bisection, there is a unique $\gamma_U \in U$ satisfying $d(\gamma_U) = x$. To prove that such an element is indeed uniquely determined by the germ, suppose that we have $[U, x] = [V, x]$ for $U, V \in \mathcal{S}$. Then, there is some idempotent $E \in E(\mathcal{S})$ with $x \in D_E = E$ and $UE = VE$. But $d(\gamma_U) = x \in E$ and so $\gamma_U \in UE = VE$, and thus $\gamma_U \in V$. This lets us define a well-defined map $\phi : \mathcal{S} \ltimes \mathcal{G}^{(0)} \to \mathcal{G}$ given by $\phi([U, x]) = \gamma_U$, where γ_U is defined as above. In other words,

$$\phi([U, x]) = d|_U^{-1}(x).$$

We now show that ϕ is a groupoid homeomorphism. Toward showing ϕ is injective, suppose $\phi([U, x]) = \phi(V, y)$. Then $d|_U^{-1}(x) = d|_V^{-1}(y)$. Let this common value be denoted z. Then $z \in U \cap V$ and $d(z) = x = y$. We use (BW2) to find some

$$W \subseteq U \cap V \subseteq \mathcal{S}$$

containing z. We can describe W in terms of the natural partial order on $\mathcal{S}$ given by set inclusion. In this sense, $W = UW^*W$ and $W = VW^*W$, since W is contained in $U \cap V$. By definition of germ equivalence, this means $[U, x] = [V, y]$.

Surjectivity follows from (BW1): if $\gamma \in \mathcal{G}$, then there exists $U \in \mathcal{S}$ such that $\gamma \in U$. Taking $[U, d(\gamma)] \in \mathcal{S} \ltimes \mathcal{G}^{(0)}$, we have $\phi([U, d(\gamma)] = \gamma$ as desired. Hence, ϕ is a bijection.

Now, we show that ϕ is a homeomorphism - we begin by showing that ϕ is continuous. Notice that ϕ is a restriction of the source mapping to an open bisection, which is an open map on its domain. Hence, ϕ is continuous.

To see that ϕ^{-1} is continuous, let U be open in $\mathcal{G}$ and take $x \in U$. Then $\phi^{-1}(x) = [U, d(x)]$, and so $\phi^{-1}(U) = \Theta_U$. Furthermore, $d(\Theta_U) = d(U)$ - that is, the composition $d|_{\Theta_U} \circ \phi^{-1}$ maps U to $d(U)$, so is nothing more than the source mapping, which is continuous. But Θ_U is a bisection, and so $d|_{\Theta_U}$ is continuous, requiring ϕ^{-1} to be continuous. Thus, ϕ is a homeomorphism.

It remains to prove that ϕ is a groupoid homomorphism. So, let $([U, x], [V, y]) \in (\mathcal{S} \ltimes \mathcal{G}^{(0)})^{(2)}$, and consider $\phi([U, x][V, y]) = \phi([UV, y]$. Then, there is some unique element $\gamma_{UV} \in UV$ such that $s(\gamma_{UV}) = y$. Write $\gamma_{UV} = \gamma_U \gamma_V$ for some $\gamma_U \in U$ and $\gamma_V \in V$. Notice that $s(\gamma_V) = s(\gamma_U \gamma_V) = y$, and so $\phi([V, y]) = \gamma_V$. Further, $s(\gamma_U) = r(\gamma_V) = x$, and so $\phi([U, x]) = \gamma_U$. Therefore,

$$\phi([U, x][V, y]) = \gamma_{UV} = \gamma_U \gamma_V = \phi([U, x])\phi([V, y]).$$

Next, consider $\phi([U, x]^{-1}) = \phi([U^{-1}, \theta_U(x)])$. Let γ_U be the unique element of U such that $s(\gamma_U) = x$. Then, $\theta_U(x) = r(\gamma_u)$, and so

$$\phi([U^{-1}, r(\gamma_u)]) = d^{-1}|_{U^{-1}}(r(\gamma_U)).$$

Since $\gamma_U \in U$, we have $\gamma_U^{-1} \in U^{-1}$ with $d(\gamma_u^{-1}) = r(\gamma_u)$. That is,

$$\phi([U^{-1}, r(\gamma_U)]) = \gamma_U^{-1} = \phi([U, x])^{-1}.$$

We have shown that ϕ preserves products and inverses, so is a groupoid homomorphism. Furthermore, ϕ is a bijective homeomorphism, so therefore a groupoid homeomorphism.

Conversely, suppose that ϕ is an isomorphism between $(\mathcal{S} \ltimes \mathcal{G}^{(0)})$ and $\mathcal{G}$. We prove that $\mathcal{S} \subseteq \mathrm{Bis}(\mathcal{G})$ is bisection wide. Let $\gamma \in \mathcal{G}$. Then, since ϕ is an isomorphism and in particular is surjective, there exists $[U, x] \in \mathcal{S} \ltimes \mathcal{G}^{(0)}$ such that $\phi([U, x]) = \gamma$. This implies $\gamma \in U$, and so (BW1) holds.

Now, suppose $\gamma \in \mathcal{G}$ such that $\gamma \in U \cap V$ where $U, V \in \mathcal{S}$. We find some $W \in \mathcal{S}$ with $\gamma \in W \subseteq U \cap V$. Since $\gamma \in U \cap V$, there is some $[U, x], [V, y] \in \mathcal{G}^{(0)} \times \mathcal{S}$ such that $\phi([U, x]) = \phi([V, y]) = \gamma$. But ϕ is injective, and so $[U, x] = [V, y]$. This means $x = y$, and there exists $E \in E(\mathcal{S})$ such that $x \in E$ and $UE = VE$. Define $W = UE = VE$. Then $W \subseteq U \cap V$, and $\gamma \in W$, being of the form $\gamma = \gamma d(\gamma)$, where $\gamma \in U$ and $d(\gamma) = x \in E$. Thus, W is our required set to satisfy (BW2), and so $\mathcal{S}$ is bisection wide. $\square$

Chapter 4

Ample Groupoids Associated to Inverse Semigroups

We have studied the action of inverse semigroups on secondary spaces, and so it is natural to ask if there exists any action intrinsic to the inverse semigroup, without the need to introduce additional data. In this chapter, we investigate two such constructions. The first is the universal groupoid $\mathcal{G}_u(\mathcal{S})$ of an inverse semigroup, the research of which was initiated by Paterson [Pat99], and as a result is often known by the name of Paterson's groupoid. The universal groupoid is built upon the spectrum $\widehat{E(\mathcal{S})}$ of $\mathcal{S}$, which is the collection of all non-zero semigroup homomorphisms (often called *semicharacters*) from $E(\mathcal{S})$ to the two-element semigroup $\{0, 1\}$. One can view $\{0, 1\}$ as other structures, such as monoids or Boolean algebras, and by putting correspondingly stricter conditions on the homomorphisms, we acquire interesting subgroupoids. We then investigate an alternate construction of the spectrum, in which it is comprised of filters, and show that the filter and semicharacter approaches are equivalent. Since Paterson's initial exposition on this groupoid, it has attracted attention from many others - we direct the reader to [KM21, EP16, Law20, MR10] for further reading.

The second construction is what we call the groupoid of ultragerms of an inverse semigroup, following [ACaH+22], where it is constructed on the Stone space of $E(\mathcal{S})$, which consists of all non-trivial ultrafilters on the idempotent set. The groupoid of ultragerms also naturally arises in the case that $\mathcal{S}$ is a Boolean inverse semigroup, in which case we can identify ultrafilters on $E(\mathcal{S})$ with non-zero Boolean homomorphisms from $E(\mathcal{S})$ to the generalized Boolean algebra $\{0, 1\}$. Literature regarding the groupoid of ultragerms is less standardized, but some further details can be found in [Ste23, Ste10].

Our discussion of groupoids that can arise from inverse semigroups via intrinsic actions is by no means exhaustive - for instance, Exel's tight groupoid [Exe08] is constructed in a similar way from *tight characters*, which coincide with the topological closure of the set of non-zero Boolean homomorphisms in $\widehat{E(\mathcal{S})}$. Note that Exel's notion of "ultracharacter", as well as that which we touch on later, is equivalent to our notion of Boolean homomorphism. Substantial effort has also been put into describing groupoids of filters, the elements of which are filters on the entire inverse semigroup as a partially ordered set. We direct the reader towards [ACaH+22, LMS13, LL13].

The aim of the later sections of this chapter is to establish a generalization of the bisection wide condition we introduced in the previous chapter. In [BM23], the term "wide" is used in the context of the inverse semigroup of open bisections of an étale groupoid to describe the conditions under which a sub-semigroup of open bisections is big enough to recover the original groupoid via its action on the unit space. A natural question to ask is whether there is a corresponding notion of bisection wide for general inverse semigroups, such that one might call a sub-semigroup $\mathcal{W}$ of $\mathcal{S}$ wide if the restriction of some intrinsic action of $\mathcal{S}$ to $\mathcal{W}$ yields the same groupoid of germs as that of $\mathcal{S}$. In this chapter, we establish such conditions for both the universal groupoid, and the groupoid of ultragerms.

4.1 Filters and Characters

4.1.1 Preliminaries

Throughout, we write $X \subseteq_{\text{fin}} Y$ to mean that X is a *finite subset* of Y. If X is a topological space, and U is an open subset of X, then we write $U \subseteq_{\circ} X$. Given a partially ordered set E, for any $F \subseteq E$, the set

$$F^{\uparrow} := \{t \in E : \exists e \in F(t \geq e)\}$$

is called the *upward-closure of* F. If $F = F^{\uparrow}$ - that is, if F is equal to its upward-closure - then we say F is *upward-closed*. In the case that F is a singleton, say $F = \{e\}$ for some $e \in E$, then F is called the *principal upward-closed set generated by* e, and we write $e^{\uparrow}$. An analogue of this holds for downward-closed sets. An *upward-directed set* (respectively, a *downward-directed set*) is a subset $X \subseteq E$ such that if $s, t \in X$ then there exists $u \in X$ satisfying $u \geq s, t$ (respectively, $u \leq s, t$).

Let E be a meet semilattice. A *semicharacter* of E is a non-zero semigroup homomorphism $\varphi : E \to \{0, 1\}$. We note that if E has a 0, then there is no obligation for a semicharacter φ to satisfy $\varphi(0) = 0$. A *character* of E is a non-zero monoid homomorphism $\varphi : E \to \{0, 1\}$. That is, characters must preserve the 0, if E contains one. Lastly, if E is a generalized Boolean algebra, a *Boolean character* is a non-zero Boolean homomorphism $E \to \{0, 1\}$, where $\{0, 1\}$ is taken as the 2-element Boolean algebra. In particular, Boolean characters must preserve joins, meets, relative complements and the 0 element. We denote the collection of semicharacters on E by $\widehat{E}$, the collection of characters on E by $\widehat{E_0}$, and the collection of Boolean characters by $\text{Spec}(E)$.

Remark 4.1.1. It is at first unclear whether semicharacters or characters are the "correct" collections to consider. In Paterson's original monograph [Pat99], he restricts his attention to semicharacters. When these collections form the unit spaces of groupoids, the difference between the two spaces amounts to a single isolated point with isotropy. We restrict our focus generally to $\widehat{E(\mathcal{S})}$, as it is the larger space, and because it connects in a natural manner to C^*-algebras of groupoids.

It is often convenient to describe characters and the structure of semilattices through the lens of filters. The following details can be found in any reasonable reference to universal algebra or lattice theory, such as [BBS84]. If P is a partially ordered set, then a *filter on* P is a subset $F \subseteq P$ such that

(F1) F is non-empty.

(F2) F is upward-closed.

(F3) F is downward directed.

A *principal filter* on E is a filter of the form $e^{\uparrow}$ for some $e \in E$. A *proper filter* is a filter which is properly contained in E. Moreover, *ultrafilters* are maximal proper filters, in that they are contained in no other proper filter. We denote by $\mathcal{L}(E)$, $\mathcal{F}(E)$, $\mathcal{F}_P(E)$, and $\mathcal{U}(E)$, the collection of all filters, proper filters, principal filters and ultrafilters on E, respectively.

Collection	**Notation**
Filters	$\mathcal{L}(E)$
Proper Filters	$\mathcal{F}(E)$
Principal Filters	$\mathcal{F}_P(E)$
Ultrafilters	$\mathcal{U}(E)$

4.1.2 Correspondence Between Filters and Characters

The following relationships between filters and characters are established (see, for instance, [Ste10]), but we provide details.

Lemma 4.1.2. *Let E be a meet semilattice, and take $\varphi \in \widehat{E}$. Then $\varphi^{-1}(1)$ is a filter on E. Conversely, if F is a filter, then the characteristic function χ_F of F is a semicharacter on E.*

Proof. Let $\varphi \in \widehat{E}$, and define $F := \varphi^{-1}(1)$. Since φ is non-zero, F is non-empty, and satisfies (F1). Take $s \in F$ such that $\varphi(s) = 1$. Then, for every $t \in E$ such that $t \geq s$, we have $\varphi(t) = \varphi(s)\varphi(t) = \varphi(st) = \varphi(s)$, since $s = st$. Hence, $\varphi(t) = 1$. This implies $t \in F$, and so F is upward-closed. Lastly, if $s, t \in F$, then $\varphi(s) = \varphi(t) = 1$, and so $\varphi(st) = \varphi(s)\varphi(t) = 1$. That is, $st \in F$, and $st \leq s, t$, giving us that F is downward directed.

Conversely, let F be a filter on E, and take $e, f \in E$. We consider a number of cases. If $e, f \in F$ then by (F3) there exists $x \in F$ such that $x \leq e, f$. But ef is the infimum of e and f, so $ef \geq x$, and so by (F2), $ef \in F$. Hence, $\chi_F(ef) = \chi_F(e)\chi_F(f) = 1$. Next, if $e, f \notin F$ then $ef \notin F$ - if it were, then since $e, f \geq ef$, (F2) would give $e, f \in F$. Thus, $\chi_F(ef) = \chi_F(e)\chi_F(f) = 0$. Lastly, without loss of generality, suppose $e \in F$ and $f \notin F$. Then, following the same argument as before, we have $ef \notin F$, and so $\chi_F(ef) = \chi_F(e)\chi_F(f) = 0$. Thus, we have shown χ_F is a semigroup homomorphism. □

Throughout, for any filter $F \subseteq E(\mathcal{S})$, we denote by φ_F the semicharacter $\varphi_F := \chi_F$. In this way, we may switch between using semicharacters and filters without ambiguity. In many situations, using one over the other is more natural. We now show an analogous result holds for proper filters and characters.

Corollary 4.1.3. Let E be a meet semilattice, and take $\varphi \in \widehat{E_0}$. Then $\varphi^{-1}(1)$ is a proper filter on E. Conversely, if F is a proper filter, then χ_F is a character on E.

Proof. Let $\varphi \in \widehat{E_0}$, and define $F := \varphi^{-1}(1)$. The fact that F is a non-empty filter follows from the previous proof. Notice that $\varphi(0) = 0$, since φ is a monoid homomorphism, and therefore 0-preserving. Thus, $0 \notin F$, and so F is proper.

Conversely, let F be a proper filter on E. It follows from an identical argument as in Lemma 4.1.2 that $\varphi := \chi_F$ is a semicharacter, and so it remains to show it is 0-preserving. Since F is proper, we have $\chi_F(x) = 0$ for some $x \in E$. It follows from (F2) that we must also have $\chi_F(y) = 0$ for all $y \leq x$. In particular, $\chi_F(0) = 0$, since $0 \leq y$ for all $y \in E$. Hence, $\varphi(0) = 0$. □

As above, if φ is a character on $E(\mathcal{S})$ then we write F_φ to denote the proper filter $F_\varphi := \varphi^{-1}(1)$. Lastly, we have a correspondence between ultrafilters and Boolean homomorphisms, which we state below. We direct the reader to [Ste10, Proposition 2.7(2)] for details.

Proposition 4.1.4. [Ste10, Proposition 2.7(2)] Let $\mathcal{S}$ be a Boolean inverse semigroup, and take a map $\varphi : E(\mathcal{S}) \to \{0, 1\}$. Then φ is a morphism of generalized Boolean algebras if and only if $\varphi^{-1}(1)$ is an ultrafilter on $E(\mathcal{S})$.

If $U \subseteq E(\mathcal{S})$ is an ultrafilter corresponding to a Boolean character φ, we write $U = U_\varphi$ and $\varphi = \varphi_U$. Lastly, if $e^\uparrow$ is a principal filter on E, then $\varphi_e^\uparrow := \chi_F$ is called a *principal semicharacter*.

Next we show that filters are preserved under semilattice isomorphisms.

Remark 4.1.5. In the following proof, we use the fact that a semilattice homomorphism is neceesarily a monotone map, and therefore a semilattice isomorphism is necessarily an order isomorphism.

Lemma 4.1.6. *Let X, Y be semilattices, and $f : X \to Y$ a semilattice isomorphism. If $F \subseteq X$ is a filter, then $f(F)$ is a filter on Y. If $F \subseteq F'$ are filters on X, then $f(F) \subseteq f(F')$, and this inclusion is strict.*

Proof. Since F is a filter, it is non-empty, and therefore $f(F)$ is also non-empty. Suppose $x \in f(F)$ and $y \geq x$. Since f is surjective, there exists some $w \in X$ such that $f(w) = y$. Then $f(w) \geq x$ and so $w \geq f^{-1}(x)$, but $f^{-1}(x) \in F$ and so $w \in F$. Hence, $f(w) = y \in f(F)$, and so $f(F)$ is upward-closed. Now, suppose $x, y \in f(F)$ - we find some $z \in f(F)$ such that $z \leq x, y$. We have that $f^{-1}(x), f^{-1}(y) \in F$, and since F is a filter, there exists $z \in F$ such that $z \leq f^{-1}(x), f^{-1}(y)$. Then $f(z) \in F$, and since

f is order-preserving, we have $f(z) \leq x, y$, and thus $f(F)$ is downward-directed.

If $e \in f(F)$ then since f is an isomorphism, there exists $y \in X$ such that $f(y) = e$. That is, $f(y) \in F(F)$, giving $y \in F$. Since $F \subseteq F'$, we have $y \in F'$, and so $e = f(y) \in f(F')$. That is, $f(F) \subseteq f(F')$. It is straightforward to show that this holds when the inclusion is strict. □

4.2 The Universal Groupoid

Similarly to how we constructed a groupoid of germs when studying the action of an inverse semigroup on a locally compact Hausdorff space, we construct the groupoid of germs of the action of an inverse semigroup $\mathcal{S}$ on its spectrum $\widehat{E(\mathcal{S})}$. This is the *universal groupoid of* $\mathcal{S}$, and is denoted $\mathcal{G}_u(\mathcal{S})$, although it is often referred to as *Paterson's groupoid*, in lieu of its first introduction by Paterson [Pat99]. We begin with some motivation, by discussing Paterson's approach to the universal groupoid via the class of $\mathcal{S}$-groupoids. Later, we will see exactly what the "universal" property of the universal groupoid is.

4.2.1 $\mathcal{S}$-Groupoids

The following approach to the universal groupoid of an inverse semigroup is that of Paterson [Pat99]. Recall that a topological groupoid is *ample* if it admits a basis of compact, open bisections. In particular, all ample groupoids are étale.

Let $\mathcal{G}$ be an ample groupoid, and let $\mathcal{S}$ be an arbitrary inverse semigroup with an inverse semigroup homomorphism $\psi : \mathcal{S} \to \mathrm{Bis}_c(\mathcal{G})$, where $\mathrm{Bis}_c(\mathcal{G})$ denotes the set of compact, open bisections on $\mathcal{G}$. Since $\mathcal{G}$ is ample, the collection $\mathrm{Bis}_c(\mathcal{G})$ forms a basis for $\mathcal{G}$.

We say that $\mathcal{G}$ is an $\mathcal{S}$-*groupoid* if the following conditions hold.

(i) $\bigcup_{s\in\mathcal{S}} \psi(s) = \mathcal{G}$.

(ii) The collection of sets $\{U_{e;f_1,\ldots,f_n}\}_{e,f_1,\ldots,f_n\in E(\mathcal{S})}$ given by

$$U_{e;f_1,\ldots,f_n} \coloneqq \psi(e)\cap\psi(f_1)^c\cap\ldots\cap\psi(f_n)^c$$

is a basis for the topology on $\mathcal{G}^{(0)}$.

The second condition defines what is often called the *patch topology*, which we will later generalize to arbitrary étale groupoids.

If $\mathcal{G}$ is an $\mathcal{S}$-groupoid, with associated homomorphism $\psi : \mathcal{S} \to \mathrm{Bis}_c(\mathcal{G})$, then we denote this pair as $(\mathcal{G}, \psi)$. In the following sections, we construct the universal groupoid $\mathcal{G}_u(\mathcal{S})$ of an inverse semigroup $\mathcal{S}$ which, as indicated by its name, is universal among all $\mathcal{S}$-groupoids, in a sense that we will formalize in a later section.

4.2.2 The Topologies on $\widehat{E(\mathcal{S})}$

For an inverse semigroup $\mathcal{S}$, there is a choice to be made regarding the topology with which to endow the semicharacter space $\widehat{E(\mathcal{S})}$. Note that we can express $\widehat{E(\mathcal{S})}$ as

$$\widehat{E(\mathcal{S})} \subseteq \{0,1\}^{E(\mathcal{S})} = \prod_{e\in E(\mathcal{S})}\{0,1\}.$$

Remark 4.2.1. Recall that $\{0,1\}^{E(\mathcal{S})}$ denotes the collection of functions from $E(\mathcal{S}) \to \{0,1\}$. We can identify this with $\prod_{e\in E(\mathcal{S})}\{0,1\}$. Let δ_e be the sequence with a 1 in the position corresponding to e, and a 0 elsewhere. An arbitrary element of $\prod_{e\in E(\mathcal{S})}$ may be written as

$$\sum_{e\in F\subseteq E(\mathcal{S})} \delta_e,$$

by which we denote the pointwise sum of sequences. We can then identify a semicharacter $\varphi : E(\mathcal{S}) \to \{0,1\}$ with the sequence

$$\sum_{e\in\varphi^{-1}(1)} \delta_e.$$

Some authors give the spectrum the relative topology inherited from the product space $\{0,1\}^{E(\mathcal{S})}$ (see [Exe08, Ste10]), while others use the topology generated by the collection of open sets $D_e := \{\varphi \in \widehat{E(\mathcal{S})} : e \in E(\mathcal{S}), \varphi(e) = 1\}$ [BEM12]. This is equivalent to letting $D_e := \{F \in \mathcal{L}(E(\mathcal{S})) : e \in F\}$. Unless otherwise specified, we will always work with the former "inherited product" topology. One notable difference of these topologies is that the spectrum equipped with the product topology is generally Hausdorff, whereas with topology generated by the sets D_e, it is almost never Hausdorff. In fact, it is only Hausdorff in the case that $E(\mathcal{S})$ has a single idempotent 1, in which case $\widehat{E(\mathcal{S})} = \{\{1\}\}$, and so is trivially Hausdorff [BEM12, p. 5].

Henceforth, we denote the collection of sets $\{D_e : e \in E(\mathcal{S})\}$ by $\mathcal{D}(E(\mathcal{S}))$, and the finer patch topology collection $\{U_{e;X} : e \in E(\mathcal{S}), X \subseteq_{fin} E(\mathcal{S})\}$ by $\mathrm{Patch}(E(\mathcal{S}))$.

Remark 4.2.2. Recall that if X is a topological space, and I a directed set, then a *net* is a function $\phi : I \to X$. A function $f : X \to Y$ is continuous if and only if for every net $(x_i)_{i\in I}$ that converges to x in X, the net $f((x_i)_{i\in I})$ converges to $f(x)$ in Y. We use this fact without further justification - see [Wil04, Theorem 11.8] for more details.

Moreover, we often speak of the inherited product topology on $\widehat{E(\mathcal{S})}$ being that of pointwise convergence. If $\prod_i X_i$ is a product space, then a net $(x_i) \subseteq \prod_i X_i$ converges to $x \in \prod_i X_i$ if and only if each of its projections $p_j(x_i)$ converge in the space X_j to $p_j(x)$ (see, for instance, [Mun03, Theorem 19.6]).

The following result is briefly mentioned by Exel in [Exe08, p. 40], but we provide a proof.

Lemma 4.2.3. *Let $\mathcal{S}$ be an inverse semigroup, and let $\widehat{E(\mathcal{S})}$ be given the inherited product topology. Then, for each idempotent e, the set D_e is clopen and compact.*

Proof. Notice that the topology here is identical to the topology of pointwise convergence, since $\{0,1\}$ is discrete. We claim that, for a fixed idempotent e, the evaluation map $\varepsilon_e : \widehat{E(\mathcal{S})} \to \{0,1\}$ given by $\varphi \mapsto \varphi(e)$ is continuous. To see this, let I be a directed set, and consider a net of characters $(\varphi_i)_{i\in I}$. Suppose $\varphi_i \to \varphi$, for some $\varphi \in \widehat{E(\mathcal{S})}$. Considering the net $(\epsilon_e(\varphi_i))_{i\in I}$, we have

$$\epsilon_e(\varphi_i) = \varphi_i(e),$$

which must converge to $\varphi(e)$, since the topology on $\widehat{E(\mathcal{S})}$ is that of pointwise convergence. Hence, ε_e is continuous, and since $\{0,1\}$ is discrete, both $\varepsilon_e^{-1}(1)$ and $\varepsilon_e^{-1}(0)$ are clopen. Hence, $D_e = \varepsilon_e^{-1}(1)$ is clopen. Lastly, Tychonoff's theorem asserts that $\{0,1\}^E$ is compact, and so D_e is compact as a closed subset of $\{0,1\}^E$. □

Lemma 4.2.4. *Let $\mathcal{S}$ be an inverse semigroup. Then the space of characters $\widehat{E_0(\mathcal{S})}$ is clopen in $\widehat{E(\mathcal{S})}$.*

Proof. Notice that we can write $\widehat{E_0(\mathcal{S})} = \widehat{E(\mathcal{S})} \setminus \{\varphi_0\}$, where φ_0 is the trivial filter on $E(\mathcal{S})$. But $D_0 = \{\varphi_0\}$, and so $\{\varphi_0\}$ is clopen, by the previous lemma. Hence $\widehat{E_0(\mathcal{S})}$ is itself clopen, being the complement of a clopen set. □

Under the assumption that $E(\mathcal{S})$ is finite, the semicharacter space $\widehat{E(\mathcal{S})}$ is discrete with respect to the topology inherited from $\{0,1\}^{E(\mathcal{S})}$ - this follows from the fact that a finite product of discrete spaces is discrete.

Remark 4.2.5. As shown by [Ste10, Proposition 2.5], the condition of finiteness on E here is too strong. We can afford simply to require all principal downward-closed sets be finite.

In general, the collection $\mathcal{D}(\mathcal{E}(\mathcal{S}))$ does not form a basis for the inherited product topology on $\widehat{E(\mathcal{S})}$. In the previous section, we introduced the *patch topology* $\mathrm{Patch}(E(\mathcal{S}))$, which consists of sets of the form

$$U_{e;X} = U_{e;f_1,\dots,f_n} := D_e \cap D_{f_1}^c \cap \dots \cap D_{f_n}^c$$

for idempotents $e \in E(\mathcal{S})$ and $f_1, \ldots, f_n \in X \subseteq_{fin} E(\mathcal{S})$. That is, $\varphi \in U_{e;f,f_1,\ldots,f_n}$ if and only if $\varphi(e) = 1$ and $\varphi(f_1) = \ldots = \varphi(f_n) = 0$. In [Pat99, p. 174], Paterson shows that $\mathrm{Patch}(E(\mathcal{S}))$ forms a basis for the product topology - we refer the reader to the discussion following [Pat99, Definition 4.3.1] for the details. Further discussion of the patch topology, can also be found in [ACaH+22, Section 2.3].

Lemma 4.2.6. *Let $\mathcal{S}$ be an inverse semigroup, and let $e \in E(\mathcal{S})$ and $X \subseteq_{fin} \mathcal{S}$. Then there exists a set X' such that $x \leq e$ for all $x \in X'$, and $U_{e;X} = U_{e;X'}$.*

Proof. Let $\varphi \in U_{e;X}$. We may assume that φ preserves 0 - if not, then $\varphi(0) = 1$, and so $\varphi(e) = 1$ for all $e \in E(\mathcal{S})$, since $e \geq 0$ by definition of 0. In this case, $X = \emptyset$, and so we set $X' = \emptyset$.

Assume $\varphi \in \widehat{E_0(\mathcal{S})}$. We obtain X' by considering two cases. Let $x \in X$.

If $x \perp e$, we can omit x, since if $\varphi \in D_e$ then $\varphi(x) = \varphi(e)\varphi(x) = \varphi(ex) = \varphi(0) = 0$, as we assumed that φ is non-trivial. That is, $D_e \subseteq D_x^c$, and thus $D_e \cap D_x^c = D_e$.

If $x \Cap e$, then $xe \neq 0$. Then, for $\varphi \in D_e$, we have $\varphi(xe) = \varphi(x)\varphi(e) = \varphi(x)$ and so we add xe to X'. In particular, we have $D_{ex} \subseteq D_x$. Note that the cases where $x \geq e$ or $x \leq e$ are just instances of having $x \Cap e$ - in that case, we have $xe = e$ and $xe = x$, respectively.

Now, X' is a finite subset of $\mathcal{S}$ such that $x \leq e$ for all $x \in X'$, and $U_{e;X} = U_{e;X'}$. □

Recall that each set D_e is clopen and compact. In the following lemma, we show that each set $U_{e;X} \in \mathrm{Patch}(E(\mathcal{S}))$ is also clopen and compact.

Lemma 4.2.7. *Let $\mathcal{S}$ be an inverse semigroup, and let $U_{e;X} \in \mathrm{Patch}(E(\mathcal{S}))$. Then $U_{e;X}$ is clopen and compact.*

Proof. The set $U_{e;X}$ is defined as

$$D_e \cap \bigcap_{x \in X} D_x^c.$$

Each D_i is clopen and compact, and so each D_i^c is clopen. Hence, $U_{e;X}$ is a finite intersection of open sets, and so is open. Furthermore, it is an intersection of closed sets, and so is closed [Mun03, Theorem 17.1]. Lastly, $U_{e;X} \subseteq D_e$, and D_e is compact, and so $U_{e;X}$ is a closed subset of a compact set and is hence compact [Mun03, Theorem 26.2]. □

4.2.3 The Spectrum Action

For an inverse semigroup $\mathcal{S}$, we define a natural action of $\mathcal{S}$ on $\widehat{E(\mathcal{S})}$. For $s \in \mathcal{S}$, recall that we have

$$D_{s^*s} = \{\varphi \in \widehat{E(\mathcal{S})} : \varphi(s^*s) = 1\}.$$

Taking $\varphi \in D_{s^*s}$, define $\theta_s(\varphi) : E(\mathcal{S}) \to \{0, 1\}$ by

$$\theta_s(\varphi)(e) = \varphi(s^*es). \tag{4.1}$$

We check below that $\theta_s(\varphi)$ is a non-zero semicharacter on $E(\mathcal{S})$ and is in D_{ss^*}. Furthermore, we claim that this defines a semigroup action on $\widehat{E(\mathcal{S})}$.

Lemma 4.2.8. *Let $\mathcal{S}$ be an inverse semigroup, and let $s \in \mathcal{S}$ and $\varphi \in D_{s^*s}$. Then $\theta_s(\varphi)$ is a non-zero semicharacter on $E(\mathcal{S})$ and is in D_{ss^*}. Furthermore, θ is a semigroup action on $\widehat{E(\mathcal{S})}$.*

Proof. First, we verify that $\theta_s(\varphi)$ is a non-zero semicharacter. Since $\varphi \in D_{s^*s}$, we know $\varphi(s^*s) = 1$. Then,

$$\theta_s(\varphi)(ss^*) = \varphi(s^*ss^*s) = \varphi(s^*s) = 1.$$

Thus, $\theta_s(\varphi)$ is non-zero. Now, if $e, f \in E(\mathcal{S})$, we have

$$\begin{aligned} \theta_s(\varphi)(ef) &= \varphi(s^*efs) = \varphi(s^*ss^*efs) = \varphi(s^*ess^*fs) \\ &= \varphi(s^*es)\varphi(s^*fs) = \theta_s(\varphi)(e)\theta_s(\varphi)(f). \end{aligned}$$

Hence, $\theta_s(\varphi)$ is multiplicative, and so is a non-zero semicharacter. Moreover, as shown above, we have $\theta_s(\varphi)(ss^*) = 1$. Hence, $\theta_s(\varphi) \in D_{ss^*}$.

To show that θ_s is bijective, it suffices to show it has a well-defined inverse - we claim that such an inverse is given by θ_{s^*}. Indeed, if $\varphi \in D_{ss^*}$, then since $\varphi(ss^*) = 1$, we use the fact that φ is multiplicative to see that

$$(\theta_{s^*} \circ \theta_s)(\varphi)(e) = \theta_{s^*}(\varphi(s^*es)) = \varphi(ss^*ess^*) = \varphi(ss^*)\varphi(e)\varphi(ss^*) = \varphi(e),$$

and so $\theta_{s^*} \circ \theta_s$ is the identity on D_{ss^*}. Similarly, for $\varphi \in D_{s^*s}$,

$$(\theta_s \circ \theta_{s^*})(\varphi)(e) = \theta_s(\varphi(ses^*)) = \varphi(s^*ses^*s) = \varphi(s^*s)\varphi(e)\varphi(s^*s) = \varphi(e),$$

and so $\theta_s \circ \theta_{s^*}$ is the identity on D_{s^*s}. Hence, we have $\theta_s^{-1} = \theta_{s^*}$.

To see that θ_s is a homeomorphism, we lastly check that it is continuous and has continuous inverse. Let $s \in \mathcal{S}$, and for $\varphi \in \widehat{E(\mathcal{S})}$, let I be a directed set and let $(\varphi_i)_{i\in I}$ be a net of semicharacters in $\widehat{E(\mathcal{S})}$ such that $\varphi_i \to \varphi$. For any $e \in E(\mathcal{S})$, we have $\theta_s(\varphi_i)(e) = \varphi_i(s^*es)$ and so $\varphi_i(s^*es) \to \varphi(s^*es) = \theta_s(\varphi)(e)$. Hence, $\theta_s(\varphi_i) \to \theta_s(\varphi)$ and so to θ_s.

To see that θ_s^{-1} is continuous, recall that $\theta_s^{-1} = \theta_{s^*}$, and note that the above argument holds for all $s \in \mathcal{S}$ - in particular, for s^*. Hence, θ_s^{-1} is a continuous inverse to θ_s. Along with the fact that θ_s is a homomorphism, this shows that θ_s is a homeomorphism from D_{s^*s} to D_{ss^*}.

Lastly, it remains to show that θ satisfies Definition 3.2.1. We have already shown that θ_s is continuous for each $s \in \mathcal{S}$. That D_e is open is given by Lemma 4.2.3, and so we have that (A1) holds. If $\varphi \in \widehat{E(\mathcal{S})}$, then φ is non-zero so there exists an idempotent e such that $\varphi(e) = 1$. Then $\varphi \in D_e$, and so the domains of the action cover $\widehat{E(\mathcal{S})}$ as desired, giving us (A2). □

One can also consider the restriction of the action of $\mathcal{S}$ on $\widehat{E(\mathcal{S})}$ to the space of characters, $\widehat{E_0(\mathcal{S})}$, and the space of Boolean characters, $\widehat{E_\infty(\mathcal{S})}$. The relationship between these groupoids, as well as groupoids of filters, is thoroughly studied in [ACaH+22]. We refer to the groupoid of germs of these restricted actions as the contracted universal groupoid $\mathcal{G}_0(\mathcal{S})$ (as in [SS21]) [1], and the groupoid of ultragerms $\mathcal{G}_\infty(\mathcal{S})$, respectively. The groupoid of ultragerms embeds as a subgroupoid of the contracted universal groupoid of $\mathcal{S}$, which in turn embeds as a subgroupoid of the universal groupoid $\mathcal{G}_u(\mathcal{S})$. As shown in our initial discussion of the groupoid of germs of an action, each of these groupoids is étale, and henceforth we tacitly identify their respective unit spaces with their corresponding character spaces via the map $[e, \varphi] \mapsto \varphi$ for any e with $\varphi \in D_e$. The following diagram summarizes the inclusions of the character spaces and their respective groupoids.

$$\begin{array}{ccccc}
\mathcal{U}(E(\mathcal{S})) \cong \widehat{E_\infty(\mathcal{S})} & \hookrightarrow & \mathcal{F}(E(\mathcal{S})) \cong \widehat{E_0(\mathcal{S})} & \hookrightarrow & \mathcal{L}(E(\mathcal{S})) \cong \widehat{E(\mathcal{S})} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{G}_\infty(\mathcal{S}) & \hookrightarrow & \mathcal{G}_0(\mathcal{S}) & \hookrightarrow & \mathcal{G}_u(\mathcal{S})
\end{array}$$

Figure 4.1: Inclusions of character spaces and their respective groupoids.

We have seen that $\mathrm{Patch}(E(\mathcal{S}))$ forms a basis for $\widehat{E(\mathcal{S})}$, and consists of clopen compact sets. It follows from this that $\mathcal{G}_u(\mathcal{S})$ has a basis consisting of clopen compact sets - we refer the reader to the result below for the details.

Proposition 4.2.9. [Pat99, Theorem 4.3.1] Let $\mathcal{S}$ be an inverse semigroup. Then $\mathcal{G}_u(\mathcal{S})$ is an ample groupoid.

4.2.4 The Finite Case

Given an inverse semigroup $\mathcal{S}$, the assumption that $\mathcal{S}$ has only finitely many idempotents greatly simplifies the topology of the spectrum and the universal groupoid. While our interest lies mainly in the infinite case, we will briefly cover some interesting properties that arise in the finite case.

[1] This groupoid is sometimes called the groupoid of proper germs.

Let E be a semilattice. We say that $e \in E$ is *primitive* if, for all $f \in E$ such that $f \leq e$ and $f \neq 0$, we have $f = e$. For example, if E is the semilattice of non-empty subsets of a set X, then the primitive elements of E are exactly the singletons of X.

Lemma 4.2.10. *If E is a finite semilattice, then the collection of filters coincides with that of principal filters. That is, $\mathcal{L}(E) = \mathcal{F}_P(E)$.*

Proof. Let F be a filter on E. If $F = E$, then $F = 0^{\uparrow}$, so we assume that F is proper. Since E is finite, F is certainly also finite - let $F = \{e_1, \ldots, e_n\}$ where $n < \infty$. Since E admits all finite meets, we have

$$f := \bigwedge_{1 \leq i \leq n} e_i \in E.$$

We claim that $F = f^{\uparrow}$. If $e \in F$, then $e = e_i$ for some $1 \leq i \leq n$. In this case, $f = e_1, \ldots, e_{i-1} e e_{i+1} \ldots e_n$, and so $f \leq e$, and $e \in f^{\uparrow}$.

Conversely, if $e \in f^{\uparrow}$, then $e \geq f$. If $e = f$, then clearly $e \in F$, since F admits finite meets. Otherwise, if $e > f$, then since f is an infimum, e cannot be a lower bound of F. That is, there must exist $e_i \in F$ such that $e \geq e_i$. But filters are upward-closed, and so $e \in F$. This shows that $\mathcal{L}(E) \subseteq \mathcal{F}_P(E)$. The reverse inclusion is trivial. □

It follows that if E is a finite semlattice, then the ultrafilters on E are precisely those of the form $p^{\uparrow}$, where p is primitive, as we show now.

Lemma 4.2.11. *Let E be a finite semilattice. Then*

$$\mathcal{U}(E) = \mathcal{F}_P^{prim} := \{e^{\uparrow} \in \mathcal{F}_P(E) : e \text{ is primitive}\}.$$

Proof. Since E is finite, by Lemma 4.2.10, we can assume that every ultrafilter is principal, since $\mathcal{U}(E) \subseteq \mathcal{L}(E)$. Let $e^{\uparrow} \in \mathcal{U}(E)$ be an ultrafilter, but toward a contradiction, assume e is not primitive. That is, there exists some non-zero $f \in E$ such that $f < e$. But then $f^{\uparrow}$ is a proper filter containing $e^{\uparrow}$, since if $x \geq e$ we have $x \geq f$. Hence, e must be primitive.

Conversely, suppose e is primitive but $e^{\uparrow}$ is not an ultrafilter. Then, there exists a proper filter $F \supset e^{\uparrow}$. By Lemma 4.2.10, we can assume $F = f^{\uparrow}$ for some $f \in E$. Thus, we have $f \leq e$, and since $f^{\uparrow}$ is a proper filter, we have $f \neq 0$. Furthermore, $f^{\uparrow}$ properly contains $e^{\uparrow}$, and so $f \neq e$. Hence, e cannot be primitive, and so $e^{\uparrow}$ must be an ultrafilter. □

Example 4.2.12. Consider the set $A = \{a, b, c\}$, and the collection $\mathcal{I}(A)$ of partial bijections from A to itself. Then $E(\mathcal{I}(A))$ is the set of partial identity functions on A, which we may identify with the powerset of A. The idempotents of $\mathcal{I}(A)$ form a semilattice, with the partial order being set containment and the meet operation given by set intersection.

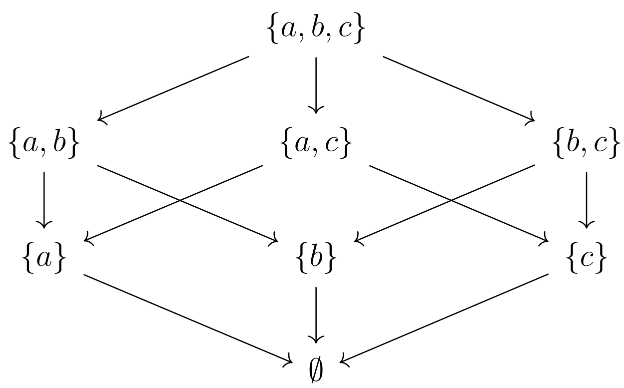


Figure 4.2: A representation of $E(\mathcal{I}(A))$ as a lattice.

The spectrum $\widehat{E(\mathcal{I}(A))}$ of this semilattice is the collection of non-zero semigroup homomorphisms $E(\mathcal{I}(A)) \to \{0, 1\}$. By Lemma 4.2.10 we know that $\widehat{E(\mathcal{I}(A))}$ is consists exactly of the principal filters on $\mathcal{I}(A)$. One example of an element of $\widehat{E(\mathcal{I}(A))}$ is seen below. Note that this is an ultrafilter, since c is primitive.

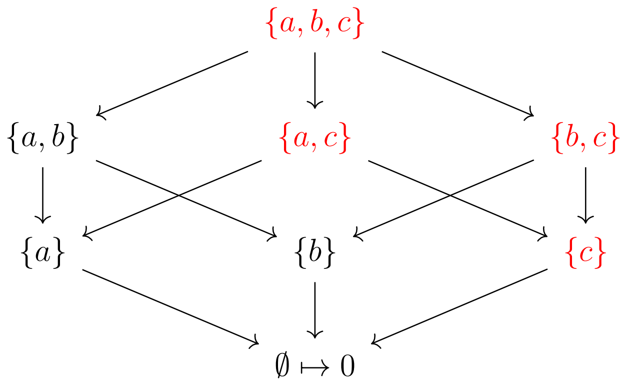


Figure 4.3: A representation of the principal filter $c^{\uparrow}$, depicted in red.

Consider the partial bijection $f : \{b, c\} \to \{a, b\}$ defined by $b \mapsto a$ and $c \mapsto b$. Then the filter $c^{\uparrow}$ shown above is in D_{f^*f} since $\{b, c\} \geq \{c\}$. We can then compute the action of f on $c^{\uparrow}$, which turns out to be the principal filter $b^{\uparrow}$.

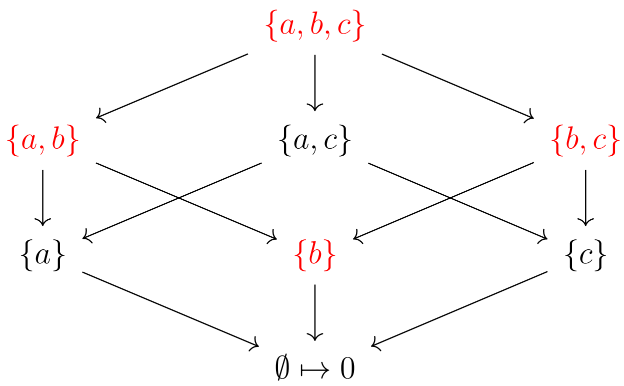


Figure 4.4: A representation of the filter $\theta_f(c^{\uparrow}) = b^{\uparrow}$, depicted in red.

Notice that $\theta_f(\varphi) \in D_{ff^*}$ since $\theta_f(\varphi)(\{a, b\}) = 1$, and also $\theta_f(\varphi)(\{b, c\}) = 1$ so $\theta_f(\varphi) \in D_{f^*f}$.

Lemma 4.2.13. *Let $\mathcal{S}$ be an inverse semigroup. If $E(\mathcal{S})$ is finite, then $\mathcal{G}_u(\mathcal{S})$ is* *discrete.*

Proof. We begin by showing that $\widehat{E(\mathcal{S})} = \mathcal{G}_u(\mathcal{S})^{(0)}$ is discrete. Recall that the topology on $\widehat{E(\mathcal{S})}$ is the relative topology inherited from the product topology on $\{0,1\}^{E(\mathcal{S})}$, where $\{0,1\}$ has the discrete topology. Since $E(\mathcal{S})$ is finite, $\{0,1\}^{E(\mathcal{S})}$ is a finite product of discrete spaces and so is discrete. Then, $\widehat{E(\mathcal{S})}$ is discrete as a subspace of a discrete space. Since $\mathcal{G}_u(\mathcal{S})$ is an étale groupoid, for each $\varphi \in \widehat{E(\mathcal{S})}$ we have that $\mathcal{G}_u(\mathcal{S})^{\varphi}$ is discrete. Furthermore,

$$\mathcal{G}_u(\mathcal{S}) = \bigcup_{\varphi \in \widehat{E(\mathcal{S})}} \mathcal{G}_u(\mathcal{S})^{\varphi}$$

is a finite union of discrete spaces, and so is discrete. □

It follows that $\mathcal{G}_u(\mathcal{S})$ is trivially Hausdorff.

Every inverse semigroup has another, more basic, intrinsic groupoid, called the *underlying groupoid* of $\mathcal{S}$. We will denote this by $\mathcal{G}_{\mathcal{S}}$. The unit space of $\mathcal{G}_{\mathcal{S}}$ is $E(\mathcal{S})$, and the range and source maps are given by $d(s) = s^*s$ and $r(s) = ss^*$. In this case, s and t are composable if and only if $s^*s = tt^*$.

The following result, briefly mentioned in [Ste10] but to which we provide details, asserts that the universal groupoid and underlying groupoid of a finite inverse semigroup coincide, up to isomorphism.

Proposition 4.2.14. [Ste10, Example 5.9] Let $\mathcal{S}$ be an inverse semigroup such that $E(\mathcal{S})$ is finite. Define a map Θ by

$$\Theta : \mathcal{G}_{\mathcal{S}} \to \mathcal{G}_u(\mathcal{S}), \quad s \mapsto [s, \varphi^{\uparrow}_{s^*s}].$$

Then Θ is an isomorphism of groupoids.

Proof. First, we check that Θ is surjective. Since $E(\mathcal{S})$ is finite, Lemma 4.2.10 states that every character on $E(\mathcal{S})$ is principal, and so is of the form $\varphi^{\uparrow}_{s^*s}$ for some $s \in \mathcal{S}$. Let $[t, \varphi^{\uparrow}_{s^*s}] \in \mathcal{G}_u(\mathcal{S})$. We claim that $\Theta(ts^*s) = [t, \varphi^{\uparrow}_{s^*s}]$.

First, notice that $\varphi^{\uparrow}_{s^*s} \in D_{t^*t}$ which implies $t^*t \geq s^*s$, and so $s^*s = s^*st^*t$. This gives us

$$\begin{aligned}(ts^*s)^*(ts^*s) &= s^*st^*ts^*s \\ &= s^*ss^*s \\ &= s^*s.\end{aligned}$$

Hence, $\varphi^{\uparrow}_{s^*s} = \varphi^{\uparrow}_{(ts^*s)^*(ts^*s)}$. Furthermore, using the above equality, we have

$$t(ts^*s)^*(ts^*s) = ts^*s,$$

and moreover,

$$(ts^*s)(ts^*s)^*(ts^*s) = (ts^*s).$$

Letting $e = (ts^*s)$, we have that $\varphi^{\uparrow}_{s^*s} \in D_e$ and $(ts^*s)e = te$, therefore $[ts^*s, \varphi^{\uparrow}_{(ts^*s)^*(ts^*s)}] = [t, \varphi^{\uparrow}_{s^*s}]$. In particular, $\Theta(ts^*s) = [t, \varphi^{\uparrow}_{s^*s}]$ as desired.

To see that Θ is injective, suppose $s, t \in \mathcal{S}$ such that $s \neq t$ and $\Theta(s) = \Theta(t)$. Then $[s, \varphi^{\uparrow}_{s^*s}] = [t, \varphi^{\uparrow}_{t^*t}]$, which implies that $\varphi^{\uparrow}_{s^*s} = \varphi^{\uparrow}_{t^*t}$. That is, if $e \in E$, then $e \geq t^*t$ if and only if $e \geq s^*s$. In particular, $s^*s \in E(\mathcal{S})$ and $s^*s \geq s^*s$, so we have $s^*s \geq t^*t$. Similarly, $t^*t \geq s^*s$, so therefore $s^*s = t^*t$. Furthermore, by germ equivalence, we know that there exists an idempotent u such that $u \geq s^*s = t^*t$ and $su = tu$. Since $u \geq s^*s = t^*t$, we have $s^*su = s^*s$ and $t^*tu = t^*t$. Then,

$$s = ss^*s = ss^*su = su = tu = tt^*tu = tt^*t = t,$$

and so Θ is bijective

We now check that Θ is a groupoid homomorphism. We begin with composability - suppose that $(s, t) \in \mathcal{G}^{(2)}_{\mathcal{S}}$; that is, $tt^* = s^*s$. We claim that $(\Theta(s), \Theta(t)) \in \mathcal{G}^{(2)}_u(\mathcal{S})$. The definition of composable pairs of elements implies that $(\Theta(s), \Theta(t)) = ([s, \varphi^{\uparrow}_{s^*s}], [t, \varphi^{\uparrow}_{t^*t}])$ is composable only when $\theta_t(\varphi^{\uparrow}_{t^*t}) = \varphi^{\uparrow}_{s^*s}$. Indeed,

$$\theta_t(\varphi^{\uparrow}_{t^*t}) = \varphi^{\uparrow}_{tt^*} = \varphi^{\uparrow}_{s^*s}.$$

This gives us that $(\Theta(s), \Theta(t)) \in \mathcal{G}_u^{(2)}(\mathcal{S})$.

Next, we verify that Θ preserves inverses and multiplication. One can see that

$$\Theta(s^*) = [s^*, \varphi^{\uparrow}_{(s^*s)^*(s^*s)}] = [s^*, \varphi^{\uparrow}_{ss^*}] = [s, \varphi^{\uparrow}_{s^*s}]^*,$$

which is precisely $\Theta(s)^{-1}$. If $s, t \in \mathcal{G}_{\mathcal{S}}$ are composable, recall that $tt^* = s^*s$. Then,

$$\Theta(st) = [st, \varphi^{\uparrow}_{(st)^*(st)}] = [st, \varphi^{\uparrow}_{t^*s^*st}] = [st, \varphi^{\uparrow}_{t^*t}] = [s, \varphi^{\uparrow}_{s^*s}][t, \varphi^{\uparrow}_{t^*t}] = \Theta(s)\Theta(t).$$

This means that Θ preserves each groupoid operation, and thus is an isomorphism. □

Note that this proposition only asserts that $\mathcal{G}_u(\mathcal{S})$ and $\mathcal{G}_{\mathcal{S}}$ are isomorphic as groupoids, but without a topology on $\mathcal{G}_{\mathcal{S}}$, there is no sense in which they might be homeomorphic. However, one might endow $\mathcal{G}_{\mathcal{S}}$ with the discrete topology, and so since $\mathcal{G}_u(\mathcal{S})$ is also discrete (see 4.2.13), the universal and underlying groupoids are also trivially homeomorphic. Hence, under the assumption of having finite idempotents, one can reduce to the underlying groupoid of $\mathcal{S}$, which in many respects is more basic than the universal groupoid.

Remark 4.2.15. In [Pat99, Proposition 4.4.6], Paterson shows that if $E(\mathcal{S})$ is not finite, the map Θ is still defined, and in fact its image is a dense subgroupoid of $\mathcal{G}_u(\mathcal{S})$.

4.2.5 Examples of Finite Inverse Semigroup Actions

The following example explicitly constructs the universal groupoid of a basic, finite, symmetric inverse semigroup.

Example 4.2.16. Let $A = \{a, b\}$ and consider $\mathcal{S} = \mathcal{I}(A)$, the set of partial bijections on A. This consists of the following maps, along with the identity maps corresponding to the idempotents.

$$s_{a,b}(a) = b, \quad s_{a,b}(b) = a,$$
$$s_a(a) = b,$$
$$s_b(b) = a,$$
$$s_0(0) = 0.$$

One can see that $\widehat{E(\mathcal{S})}$ consists of 4 characters, as $E(\mathcal{S})$ admits 4 filters, and it is straightforward to calculate the universal groupoid $\mathcal{S}$.

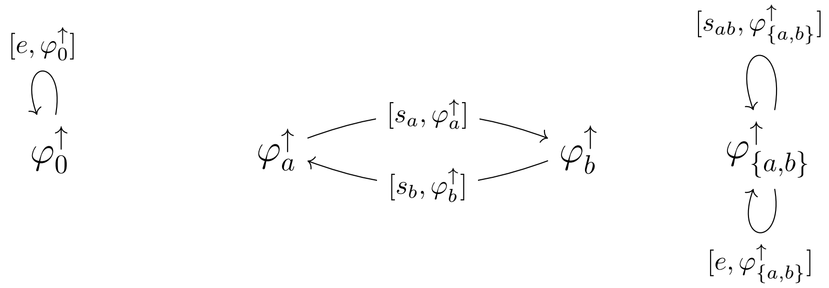


Figure 4.5: $\mathcal{G}_u(\mathcal{I}(\mathcal{S}))$

As can be seen, the groupoid consists of the group S_2 at $\varphi_{\{a,b\}}^{\uparrow}$ and then the objects φ_a and φ_b with one morphism between them. We can then reduce the universal groupoid to find the contracted universal groupoid and groupoid of ultragerms, which are seen to be subgroupoids of the universal groupoid.

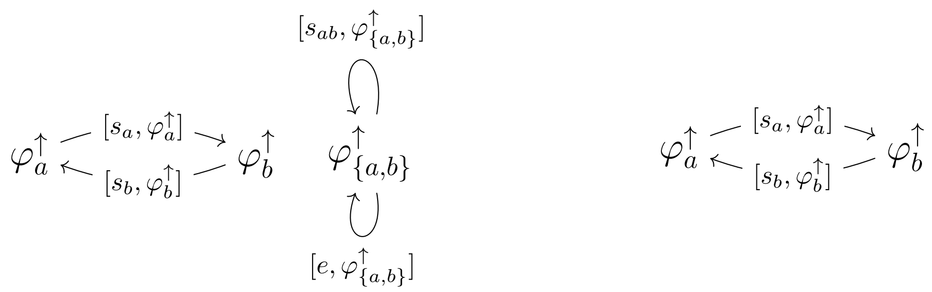


Figure 4.6: $\mathcal{G}_0(\mathcal{I}(\mathcal{S}))$ (left) and $\mathcal{G}_\infty(\mathcal{I}(\mathcal{S}))$ (right).

Remark 4.2.17. One might generalize the above by noticing that if $\mathcal{I}_n$ is the finite symmetric inverse semigroup on n elements, then the universal groupoid $\mathcal{G}_u(\mathcal{I}_n)$ will necessarily contain the symmetric group S_n at the principal filter $\varphi^{\uparrow}_{\{1,\ldots,n\}}$, as the bijections on $\{1, \ldots, n\}$ are exactly the permutations on n elements. Then, one has the trivial group at the trivial filter. Lastly, since partial bijections are between sets of equal cardinality, the remainder of the groupoid consists of $n - 1$ subgroupoids, where the i-th subgroupoid contains the principal filters generated by subsets of cardinality $1 \leq i < n$. Then, the groupoid of ultragerms with consist of the subgroupoid containing the primitive principal filters, which are those generated by singletons.

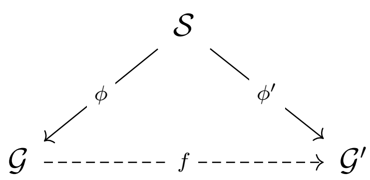
4.2.6 The Universal Groupoid is Universal

We now briefly show that $\mathcal{G}_u(\mathcal{S})$ is indeed universal, in the sense that if $\mathcal{G}'$ is any $\mathcal{S}$-groupoid, then $\mathcal{G}'$ can be associated with a closed subgroupoid of $\mathcal{G}_u(\mathcal{S})$, via some well-behaved surjective homomorphism. Furthermore, it can be shown that $\mathcal{G}_u(\mathcal{S})$ is a faithful $\mathcal{S}$-groupoid, such that $\mathcal{S} \to \mathrm{Bis}_c(\mathcal{G}_u(\mathcal{S}))$ is injective. In this sense, the universal groupoid captures all information about about the inverse semigroup $\mathcal{S}$.

Let $\mathcal{S}$ be an inverse semigroup, and let $(\mathcal{G}, \phi), (\mathcal{G}', \phi')$ be $\mathcal{S}$-groupoids with associated maps $\phi : \mathcal{S} \to \mathrm{Bis}_c(\mathcal{G})$ and $\phi' : \mathcal{S} \to \mathrm{Bis}_c(\mathcal{G}')$. A map $f : \mathcal{G} \to \mathcal{G}'$ is said to be $\mathcal{S}$-*equivariant* if

$$f \circ \phi = \phi'.$$

That is, if the follow diagram commutes.



The following can be found in [Pat99, Proposition 4.3.5], or [MR10, p. 16].

Recall that if $\mathcal{G}$ is a groupoid, and $Z \subseteq \mathcal{G}^{(0)}$, then $\mathcal{G}|_Z$ denotes the *reduction of $\mathcal{G}$ by Z*, and is given by $r^{-1}(Z) \cap d^{-1}(Z)$.

Lemma 4.2.18. *Let $\mathcal{S}$ be an inverse semigroup. Then $\mathcal{G}_u(\mathcal{S})$ is an $\mathcal{S}$-groupoid.*

Proof. Define $\phi : \mathcal{S} \to \mathrm{Bis}_c(\mathcal{G}_u(\mathcal{S}))$ by $s \mapsto \Theta(s, D_{s^*s})$. We know that each $\Theta(s, D_{s^*s})$ is an open bisection (see Proposition 3.2.8), and since the domain map is a homeomorphism on $\Theta(s, D_{s^*s})$, we have that $d(\Theta(s, D_{s^*s})) = D_{s^*s}$, which is compact, giving us $\Theta(s, D_{s^*s}) \in \mathrm{Bis}_c(\mathcal{G}_u(\mathcal{S})))$.

We check that ϕ is a homomorphism. Recall that we have

$$D_{(st)^*(st)} = \theta_{t^*}(D_{s^*s} \cap D_{tt^*}).$$

If $[st, \varphi] \in \Theta(st, D_{(st)^*(st)})$, then there exists $\psi \in D_{s^*s} \cap D_{tt^*}$ such that $\theta_t(\phi) = \psi$. Then, $[st, \varphi] = [s, \psi][t, \varphi] \in \Theta(s, D_{s^*s})\Theta(t, D_{t^*t})$.

Conversely, if $[s, \psi][t, \varphi] \in \Theta(s, D_{s^*s})\Theta(t, D_{t^*t})$, then $\theta_t(\varphi) = \psi$, and so $\psi \in D_{s^*s} \cap D_{tt^*}$, giving us

$$\varphi \in \theta_{t^*}(D_{s^*s} \cap D_{tt^*}) = D_{(st)^*(st)}.$$

Thus, $[s, \psi][t, \varphi] = [st, \varphi] \in \Theta(st, D_{(st)^*(st)})$, and we have shown equality.

To see $\phi(s^*) = \phi(s)^*$, notice that

$$\phi(s^*) = \Theta(s^*, D_{ss^*}) = \Theta(s, D_{s^*s})^*$$

as required.

By Proposition 4.2.9, the groupoid $\mathcal{G}_u(\mathcal{S})$ is ample, and as shown by Paterson ([Pat99, p. 174]), the collection $\mathrm{Patch}(E(\mathcal{S}))$ is a basis for its topology. We know that the sets $\Theta(s, D_{s^*s})$ cover $\mathcal{G}_u(\mathcal{S})$, since the collection $\mathcal{D}(E(\mathcal{S}))$ covers $\widehat{E(\mathcal{S})}$, and we have just shown that there exists a semigroup homomorphism $\phi : \mathcal{S} \to \mathrm{Bis}_c(\mathcal{G}_u(\mathcal{S}))$. Hence, $\mathcal{G}_u(\mathcal{S})$ is an $\mathcal{S}$-groupoid. □

Proposition 4.2.19. [Pat99, Proposition 4.3.5] Let $(\mathcal{G}, \phi)$ be an $\mathcal{S}$-groupoid. Then there exists a closed, invariant subset $Z \subseteq \mathcal{G}_u(\mathcal{S})^{(0)}$ such that $\mathcal{G}^{(0)} \cong Z$.

Furthermore, there exists a continuous, surjective, $\mathcal{S}$-equivariant homomorphism

$$p : \mathcal{G}_u(\mathcal{S})|_Z \to \mathcal{G}$$

such that p is a retraction onto Z - that is, $p|_Z = \mathrm{id}_Z$.

The above result demonstrates that every $\mathcal{S}$-groupoid corresponds to a subgroupoid of $\mathcal{G}_u(\mathcal{S})$ in a well-defined manner. Paterson also mentions the following result (see [Pat99, p. 180]), but omits the proof, so we provide the details.

Proposition 4.2.20. Let $\mathcal{S}$ be an inverse semigroup, and let $(\mathcal{G}_u(\mathcal{S}), \phi)$ be the universal groupoid of $\mathcal{S}$. Then $\mathcal{G}_u(\mathcal{S})$ is a faithful $\mathcal{S}$-groupoid - that is, ϕ is injective.

Proof. Toward a contradiction, suppose there exist $s \neq t \in \mathcal{S}$ such that $\phi(s) = \phi(t)$. By definition, this means $\Theta(s, D_{s^*s}) = \Theta(t, D_{t^*t})$. Hence, for every $\varphi \in D_{s^*s}$, there exists $[t, \phi] \in \Theta(t, D_{t^*t})$ such that $[s, \varphi] = [t, \phi]$. In particular, $\varphi = \phi$. This implies that $\theta_s(\varphi) = \theta_t(\varphi)$ - this holds for all such φ, and so $\theta_s = \theta_t$. But the mapping $s \mapsto \theta_s$ is a homomorphism - and so is injective - giving $s = t$. □

4.2.7 Homeomorphism of Unit Spaces

We now turn our attention back to the following question. Suppose $\mathcal{S}$ and $\mathcal{W}$ are inverse semigroups such that $\mathcal{W} \subseteq \mathcal{S}$. Under what conditions does $\mathcal{G}_u(\mathcal{S}) \cong \mathcal{G}_u(\mathcal{W})$ hold? One would expect that $\mathcal{S} \cong \mathcal{W}$ is sufficient - indeed, we show this is true at the end of this section. We begin by assuming the existence of an isomorphism between $E(\mathcal{S})$ and $E(\mathcal{W})$. This allows us to induce a homeomorphism between the respective character spaces of $\mathcal{S}$ and $\mathcal{W}$, with respect to the inherited product topology.

First, we characterize the continuous map between semicharacter spaces induced by a homomorphism between semilattices. We refer the reader to

[Rie16, Section 1.3] for some basic facts about functors.

Remark 4.2.21. Recall that if $F : C \to D$ is a functor between categories C and D, we say F is *contravariant* if it switches directions of morphisms. That is, whenever $f : x \to y$ is a morphism in C, then $F(f) : F(y) \to F(x)$ is a morphism in D [Rie16, Definition 1.3.5]. Below, we take C to be the category of meet-semilattices with semilattice homomorphisms, and D to be the category of spaces of non-zero semicharacters with continuous maps.

Lemma 4.2.22. *Let F be the functor from the categories of meet-semilattices to the category of non-zero semicharacter spaces with the inherited product topology that takes E to its semicharacter space $\widehat{E}$. Then F is a contravariant functor, and if $\iota : E \to E'$ is a semilattice homomorphism, then $F(\iota) : \widehat{E} \to \widehat{E'}$ defined by*

$$F(\iota)(\varphi)(e) = \varphi(\iota(e)) \tag{4.2}$$

for $\varphi \in \widehat{E'}$ and $e \in E$ is a continuous mapping.

Proof. One can see that, by definition, F is contravariant. We begin by ensuring that if $\iota : E \to E'$ is a semilattice homomorphism, then $F(\iota)$ is a map from $\widehat{E'}$ to $\widehat{E}$. In particular, we check that the image of $F(\iota)$ consists of non-zero semicharacters.

Let $\varphi \in \widehat{E'}$. Since φ is non-zero, there exists an idempotent $e \in E$ such that $\varphi(e) = 1$. We know ι is surjective, so there exists $f \in E'$ such that $\iota(f) = e$. Then,

$$F(\iota)(\varphi)(f) = \varphi(\iota(f)) = \varphi(e) = 1.$$

Hence, $F(\iota)(\varphi)$ is non-zero. We now check that it is multiplicative. If $e, f \in E'$, then

$$F(\iota)(\varphi)(ef) = \varphi(\iota(ef)) = \varphi(\iota(e)\iota(f)) = \varphi(\iota(e))\varphi(\iota(f)) = F(\iota)(\varphi)(e)\sigma(\varphi)(f).$$

Next, we check that if $\iota : E \to E'$ is a semilattice isomorphism, then $F(\iota) : \widehat{E'} \to \widehat{E}$ is a continuous mapping.

Let I be a directed set, and $(\varphi_i)_{i\in I}$ a net in $\widehat{E}$ such that $\varphi_i \to \varphi$ for some $\varphi \in \widehat{E}$. Since the topology on the spectrum is that of pointwise convergence (see Remark 4.2.2), $\varphi_i \to \varphi$ if and only if $\varphi_i(e) \to \varphi(e)$ for every $e \in E$. Consider the net

$$(F(\iota)(\varphi_i)(e))_{i\in I}$$

in $\{0,1\}$. This is equal to $(\varphi_i(\iota(e)))_{i\in I}$, which converges to $\varphi(\iota(e)) = F(\iota)(\varphi)$. Thus, we have $F(\iota)(\varphi_i) \to F(\iota)(\varphi)$, and so $F(\iota)$ is a continuous map.

We lastly check that F preserves identity morphisms and composition of morphisms.

First, if E is a meet-semilattice, and id_E is the identity morphism at E, then we see that

$$F(\mathrm{id}_E)(\varphi)(e) = \varphi(\mathrm{id}_E(e)) = \varphi(e).$$

Hence, F preserves identity morphisms.

Now, suppose $\iota_1 : E \to E'$ and $\iota_2 : E' \to E''$ are morphisms. We claim that $F(\iota_2)F(\iota_1) = F(\iota_2\iota_1)$. We have

$$\begin{aligned} F(\iota_2)F(\iota_1)(\varphi)(e) &= F(\iota_2)(\varphi)(\iota_1(e)) \\ &= \varphi(\iota_2(\iota_1(e))) \\ &= \varphi((\iota_2\iota_1)e) \\ &= F(\iota_2\iota_1)(\varphi)(e). \end{aligned}$$

Hence, F preserves composition and is thus a functor. □

$$\begin{array}{ccc} E & \xrightarrow{F} & \widehat{E} \\ {\scriptstyle\iota}\downarrow & & \uparrow{\scriptstyle F(\iota)} \\ E' & \xrightarrow{F} & \widehat{E'} \end{array}$$

Remark 4.2.23. By [Rie16, Lemma 1.3.8], we know that functors preserve isomorphisms. Hence, if $\iota : E \to E'$ is an isomorphism, then so is $F(\iota) : \widehat{E'} \to \widehat{E}$ with inverse $F(\iota^{-1}) : \widehat{E} \to \widehat{E'}$.

Henceforth, if $\mathcal{S}$ and $\mathcal{W}$ are inverse semigroups, and $\iota : E(\mathcal{S}) \to E(\mathcal{W})$ is a map between idempotents, we denote by σ the induced map given above by $F(\iota) : \widehat{E(\mathcal{W})} \to \widehat{E(\mathcal{S})}$.

Proposition 4.2.24. Let $\mathcal{S}$ and $\mathcal{W}$ be inverse semigroups. If $\iota : E(\mathcal{W}) \to E(\mathcal{S})$ is a semilattice isomorphism, then $\sigma : \widehat{E(\mathcal{S})} \to \widehat{E(\mathcal{W})}$ as given in Equation 4.2 is a homeomorphism.

Proof. By the remark above, we know that if $\iota : E(\mathcal{W}) \to E(\mathcal{S})$ is a semilattice isomorphism, then $\sigma : \widehat{E(\mathcal{S})} \to \widehat{E(\mathcal{W})}$ is also a continuous isomorphism of semicharacter spaces. Hence, it remains to show that σ has a continuous inverse.

The inverse σ^{-1} of σ can be defined for $\varphi \in \widehat{E(\mathcal{W})}$ and $e \in E(\mathcal{S})$ as $\sigma^{-1}(\varphi)(e) = \varphi(\iota^{-1}(e))$. The function ι^{-1} is well-defined since ι is an isomorphism. We can check that

$$(\sigma^{-1} \circ \sigma)(\varphi)(e) = \sigma^{-1}(\varphi(\iota(e))) = \varphi(\iota^{-1}(\iota(e))) = \varphi(e).$$

show that σ^{-1} is continuous. We use a similar argument as that for σ. Let I be a directed set and let $(\varphi_i)_{i \in I}$ be a net in $\widehat{E(\mathcal{W})}$ such that $\varphi_i \to \varphi$ for some $\varphi \in \widehat{E(\mathcal{W})}$. This is equivalent to having $\varphi_i(e) \to \varphi(e)$ for every $e \in E(\mathcal{W})$ (see 4.2.2). Then,

$$\sigma^{-1}(\varphi_i)(e) = \varphi_i(\iota^{-1})(e) \to \varphi(\iota^{-1})(e) = \sigma^{-1}(\varphi)(e).$$

Hence σ^{-1} is continuous, and σ is a homeomorphism. □

It may seem a trivial claim that if we begin with isomorphic inverse semigroups, that their universal groupoids are homeomorphism. This is entirely unsurprising, but non-trivial to prove.

Proposition 4.2.25. Let $\mathcal{S}$ and $\mathcal{W}$ be inverse semigroups with $\mathcal{W} \subseteq \mathcal{S}$. If there exists an isomorphism $\iota : \mathcal{S} \to \mathcal{W}$, then the map $\sigma : \mathcal{G}_u(\mathcal{S}) \to \mathcal{G}_u(\mathcal{W})$ as defined in Equation 4.3 is a groupoid homeomorphism.

Proof. Suppose there exists an isomorphism $\iota : \mathcal{S} \to \mathcal{W}$. Define $\sigma : \widehat{E(\mathcal{W})} \to \widehat{E(\mathcal{S})}$ by

$$\sigma(\varphi)(e) = \varphi(\iota(e))$$

for all $e \in E(\mathcal{S})$. We also define $\rho : \mathcal{G}_u(\mathcal{S}) \to \mathcal{G}_u(\mathcal{W})$ by

$$[s, \varphi] \mapsto [\iota(s), \sigma^{-1}(\varphi)].$$

We claim that ρ is a homeomorphism.

First, we check that ρ is well-defined. Suppose $[s, \varphi] = [t, \gamma]$, and consider $\rho([s, \varphi]) = [\iota(s), \sigma^{-1}(\varphi)]$ and $\rho([t, \gamma]) = [\iota(t), \sigma^{-1}(\gamma)]$. We have $\varphi = \gamma$, and so since σ is a homeomorphism, we have $\sigma^{-1}(\varphi) = \sigma^{-1}(\gamma)$. Furthermore, we have the existence of an idempotent $e \in E(\mathcal{S})$ such that $\varphi \in D_e$ and $se = te$. Notice that

$$\sigma^{-1}(\varphi)(\iota(e)) = \varphi(e) = 1,$$

and so $\sigma^{-1}(\varphi) \in D_{\iota(e)}$. Furthermore, $se = te$ implies $\iota(se) = \iota(te)$, but ι is an isomorphism, so $\iota(s)\iota(e) = \iota(t)\iota(e)$. Hence, we have $[\iota(s), \sigma^{-1}(\varphi)] = [\iota(t), \sigma^{-1}(\gamma)]$.

That ρ is surjective is straightforward to check. Since ι is an isomorphism, we can write a germ of $\mathcal{G}_u(\mathcal{W})$ as $[\iota(s), \varphi]$ for some $s \in \mathcal{S}$. Then $\rho([s, \sigma(\varphi)]) = [\iota(s), \varphi]$.

To see that ρ is injective, suppose that $\rho([s, \varphi]) = \rho([t, \gamma])$ - that is, $[\iota(s), \sigma^{-1}(\varphi)] = [\iota(t), \sigma^{-1}(\gamma)]$. As before, this means $\varphi = \gamma$, and we have an idempotent $\iota(f) \in E(\mathcal{W})$ such that $\sigma^{-1}(\varphi) \in D_{\iota(f)}$ and $\iota(s)\iota(f) = \iota(t)\iota(f)$. Notice that

$$\varphi(f) = \sigma^{-1}(\varphi)(\iota(f)) = 1,$$

and so $\varphi \in D_f$. Furthermore, this implies $\iota(sf) = \iota(tf)$, and ι being injective gives $sf = tf$, where $f \in E(\mathcal{S})$. Hence, $[s, \varphi] = [t, \gamma]$, and so ρ is injective.

We have shown that ρ is a bijection, so now we show that it is continuous. Let $\Theta(s, U) \subseteq \mathcal{G}_u(\mathcal{W})$ be open, where $U \subseteq_\circ D_{s^*s}$. Then, since ι is a bijection and σ is a homeomorphism,

$$\rho^{-1}(\Theta(s,U)) = \{[t,\varphi] \in \mathcal{G}_u(\mathcal{S}) : \iota(t) = s, \sigma^{-1}(\varphi) \in U\} = \Theta(\iota^{-1}(s), \sigma(U),$$

which is open in $\mathcal{G}_u(\mathcal{S})$.

The inverse of ρ can be given as

$$\rho^{-1} : \mathcal{G}_u(\mathcal{W}) \to \mathcal{G}_u(\mathcal{S}), \quad [s,\varphi] \mapsto [\iota^{-1}(s), \sigma(\varphi)].$$

Since ι is an isomorphism and σ is a homeomorphism, we have that ι^{-1} is also an isomorphism, and σ^{-1} is also a homeomorphism. Hence, the continuity of ρ^{-1} follows from an identical argument as above.

It remains to show that ρ preserves composability and is a groupoid homomorphism. Suppose $([s,\varphi],[t,\gamma]) \in \mathcal{G}_u(\mathcal{S})^{(2)}$. We claim that $(\rho([s,\varphi]), \rho([t,\gamma])) \in \mathcal{G}_u(\mathcal{W})^{(2)}$. We know that $\theta_t(\gamma) = \varphi$. Then,

$$\begin{aligned}\theta_{\iota(t)}(\sigma^{-1}(\gamma)) &= \sigma^{-1}(\gamma)(\iota(t)^*\iota(e)\iota(t)) \\ &= \sigma^{-1}(\gamma)(\iota(t^*et)) \\ &= \sigma^{-1}(\theta_t(\gamma))(e) \\ &= \sigma^{-1}(\varphi)(e).\end{aligned}$$

Hence, $(\rho([s,\varphi]), \rho([t,\gamma])) \in \mathcal{G}_u(\mathcal{W})^{(2)}$.

We check that ρ preserves multiplication. If $[s,\varphi],[t,\gamma] \in \mathcal{G}_u(\mathcal{S})$ are composable, then we have

$$\begin{aligned}\rho([s,\varphi][t,\gamma]) &= \rho([st,\gamma]) \\ &= [\iota(s)\iota(t), \sigma^{-1}(\gamma)] \\ &= [\iota(s)\iota(t), \sigma^{-1}(\gamma)] \\ &= [\iota(s), \sigma^{-1}(\varphi)][\iota(t), \sigma^{-1}(\gamma)],\end{aligned}$$

since we know that $\theta_{\iota(t)}(\sigma^{-1}(\gamma)) = \sigma^{-1}(\varphi)$. Hence, this is equal to $\rho([s,\varphi])\rho([t,\varphi])$.

Lastly, let $[s, \varphi] \in \mathcal{G}_u(\mathcal{S})$. Then,

$$\begin{aligned}\rho([s,\varphi]^{-1}) &= \rho([s^*, \theta_s(\varphi)]) \\ &= [\iota(s^*), \sigma^{-1}(\theta_s(\varphi))] \\ &= [\iota(s)^*, \theta_s(\sigma^{-1}(\varphi))] \\ &= [\iota(s), \sigma^{-1}(\varphi)]^{-1} \\ &= \rho([s,\varphi])^{-1}.\end{aligned}$$

So, we have shown that ρ is a groupoid homomorphism, and hence we have that ρ is a groupoid homeomorphism. □

4.2.8 Wideness

Following [Pat99, Definition 2.3.3], we define a *pseudogroup* on a topological space X as an inverse semigroup of homeomorphisms between open subsets of X, and denote this by $\Gamma(X)$. We write $\theta_x \in \Gamma(X)$ to say that θ_x is a homeomorphism in $\Gamma(X)$ such that $\theta_x : U \to V$ and $U, V \subseteq X$ are open.

If $\Gamma(X), \Gamma(Y)$ are pseudogroups on spaces X and Y respectively, and $f : X \to Y$ is a map, then by abuse of notation we write $\Gamma(Y)f$ to mean the collection $\{\theta_y \circ f : \theta_y \in \Gamma(Y)\}$.

Let $\mathcal{S}$ be an inverse semigroup with semicharacter space $\widehat{E(\mathcal{S})}$. Recall that the action of $\mathcal{S}$ on $\widehat{E(\mathcal{S})}$ is given by the homeomorphisms $\theta_s : D_{s^*s} \to D_{ss^*}$ for $s \in \mathcal{S}$. By $\Gamma_{\mathcal{S}}(\widehat{E(\mathcal{S})})$ we denote the *pseudogroup on $\widehat{E(\mathcal{S})}$ generated by $\mathcal{S}$*, given by the collection $\{\theta_s : s \in \mathcal{S}\}$ where each θ_s is the homeomorphism between open subsets of $\widehat{E(\mathcal{S})}$ as given by Equation 4.1.

Let $\mathcal{S}, \mathcal{W}$ be inverse semigroups with pseudogroups $\Gamma_{\mathcal{S}}(\widehat{E(\mathcal{S})}), \Gamma_{\mathcal{W}}(\widehat{E(\mathcal{W})})$ on their semicharacter spaces. If $f : X \to Y$ is a homeomorphism, then we say that $\Gamma_{\mathcal{S}}(\widehat{E(\mathcal{S})})$ and $\Gamma_{\mathcal{W}}(\widehat{E(\mathcal{W})})$ are *conjugate via f* if

$$\Gamma_{\mathcal{S}}(\widehat{E(\mathcal{S})}) = f^{-1}\Gamma_{\mathcal{W}}(\widehat{E(\mathcal{W})})f.$$

In particular, this asserts that whenever $s \in \mathcal{S}$, there exists some $w \in \mathcal{W}$ such that whenever $\varphi \in D_{s^*s}$, we have $\sigma(\varphi) \in D_{w^*w}$ and $\theta_s(\varphi) = (f^{-1} \circ \theta_w \circ f)(\varphi)$.

There may exist numerous elements in $\mathcal{W}$ satisfying this condition with respect to $s \in \mathcal{S}$ - the collection of these is denoted $\mathcal{W}_s$. That is,

$$\mathcal{W}_s := \{w \in \mathcal{W} : \theta_s = f^{-1} \circ \theta_w \circ f\}.$$

Definition 4.2.26. Let $\mathcal{S}$ be an inverse semigroup, and $\mathcal{W} \subseteq \mathcal{S}$ a subsemigroup. We say that $\mathcal{W}$ is *wide in* $\mathcal{S}$ (or just *wide* when there is no ambiguity) if the following criteria hold.

(W1) There exists an isomorphism $\iota : E(\mathcal{W}) \to E(\mathcal{S})$.

(W2) The pseudogroups $\Gamma_{\mathcal{S}}(\widehat{E(\mathcal{S})})$ and $\Gamma_{\mathcal{W}}(\widehat{E(\mathcal{W})})$ are conjugate via σ - that is,
$$\Gamma_{\mathcal{S}}(\widehat{E(\mathcal{S})}) = \sigma^{-1}\Gamma_{\mathcal{W}}(\widehat{E(\mathcal{W})})\sigma.$$

(W3) Whenever $s, t \in \mathcal{S}$ and $F \in \mathcal{F}(E(\mathcal{S}))$, there exists $e \in F$ such that $se = te$ if and only if there exists $f \in E(\mathcal{W})$ such that $\iota(f) \in F$ and $wf = vf$ for all $w \in \mathcal{W}_s$, and $v \in \mathcal{W}_t$.

Suppose $\mathcal{W}$ is wide in $\mathcal{S}$. We can now define $\rho : \mathcal{G}_u(\mathcal{S}) \to \mathcal{G}_u(\mathcal{W})$ by

$$[s, \varphi] \mapsto [w, \sigma(\varphi)] \quad \text{for any } w \in \mathcal{W}_s. \tag{4.3}$$

In the following lemma, we show that our sets $\mathcal{W}_s$ behave nicely with respect to multiplication and inversion.

Lemma 4.2.27. *Let $\mathcal{S}$ and $\mathcal{W}$ be inverse semigroups such that $\mathcal{W} \subseteq \mathcal{S}$. If $s, t \in \mathcal{S}$, then we have the following.*

(i) $\mathcal{W}_s\mathcal{W}_t \subseteq \mathcal{W}_{st}$.

(ii) $\mathcal{W}_{s^*}\mathcal{W}_s{}^*$.

Proof. Let $s, t \in \mathcal{S}$, and take $wv \in \mathcal{W}_s\mathcal{W}_t$, such that $w \in \mathcal{W}_s$ and $v \in \mathcal{W}_t$. We have

$$\begin{aligned}\sigma^{-1} \circ \theta_{wv} \circ \sigma &= \sigma^{-1} \circ \theta_w \circ \theta_v \circ \sigma \\ &= \sigma^{-1} \circ \theta_w \circ \sigma \circ \sigma^{-1} \circ \theta_v \circ \sigma \\ &= \theta_s \circ \theta_t \\ &= \theta_{st}.\end{aligned}$$

Hence, $wv \in \mathcal{W}_{st}$.

Now, we show that ${\mathcal{W}_s}^* \subseteq \mathcal{W}_{s^*}$, so let $s \in \mathcal{S}$ and $w^* \in {\mathcal{W}_s}^*$. That is, $w \in \mathcal{W}_s$. Then,

$$\begin{aligned}\sigma^{-1} \circ \theta_{w^*} \circ \sigma &= \sigma^{-1} \circ \theta_w^{-1} \circ \sigma \\ &= (\sigma^{-1} \circ \theta_w \circ \sigma)^{-1} \\ &= \theta_s^{-1} \\ &= \theta_{s^*}.\end{aligned}$$

Hence, $w^* \in \mathcal{W}_{s^*}$ as required. To see the reverse inclusion holds, take $w \in \mathcal{W}_{s^*}$. Then,

$$\begin{aligned}\sigma^{-1} \circ \theta_{w^*} \circ \sigma &= (\sigma^{-1} \circ \theta_w \circ \sigma)^{-1} \\ &= (\theta_{s^*})^{-1} \\ &= \theta_s.\end{aligned}$$

Hence, $w^* \in \mathcal{W}_s$, and so $w \in {\mathcal{W}_s}^*$ as desired. Thus, $\mathcal{W}_{s^*} = {\mathcal{W}_s}^*$. □

Remark 4.2.28. We remark that the above lemma implies if $w \in \mathcal{W}_s$, then ${\mathcal{W}_{s^*}}^* = \mathcal{W}_{s^{**}} = \mathcal{W}_s$, and so $w^* \in \mathcal{W}_{s^*}$.

Theorem 4.2.29. Let $\mathcal{S}, \mathcal{W}$ be inverse semigroups with $\mathcal{W} \subseteq \mathcal{S}$. If $\mathcal{W}$ is wide in $\mathcal{S}$, then the map ρ given by

$$\mathcal{G}_u(\mathcal{S}) \to \mathcal{G}_u(\mathcal{W}), \quad [s, \varphi] \mapsto [w, \sigma(\varphi)] \quad \text{for any } w \in \mathcal{W}_s$$

is a well-defined homeomorphism of groupoids.

Proof. Assume that $\mathcal{W}$ is wide. We begin by checking that ρ is well-defined.

We verify that if $[s, \varphi_1] = [t, \varphi_2]$ then $\rho([s, \varphi_1]) = \rho([t, \varphi_2])$. Assuming the former, we must have $\varphi := \varphi_1 = \varphi_2$, and the existence of an idempotent $e \in E(\mathcal{S})$ such that $\varphi \in D_e$, and $se = te$. We claim that $\rho([s, \varphi_1]) = \rho([t, \varphi_2])$.

Suppose $\rho([s, \varphi_1]) = [w, \sigma(\varphi_1)]$ and $\rho([t, \varphi_2]) = [v, \sigma(\varphi_2)]$, for some $w \in \mathcal{W}_s$ and $v \in \mathcal{W}_t$. Since $\mathcal{W}$ is wide, we know that $\mathcal{W}_s$ is non-empty for all $s \in \mathcal{S}$. By condition (W3), since $\varphi(e) = 1$ and $se = te$, we have the existence of an idempotent $f \in E(\mathcal{W})$ such that $\varphi(\iota(f)) = 1$ and $xf = yf$ whenever $x \in \mathcal{W}_s$ and $y \in \mathcal{W}_t$. In particular, $wf = vf$. Since $\varphi(\iota(f)) = 1$, we have $\sigma(\varphi)(f) = 1$ and so $\sigma(\varphi) \in D_f$. Hence, $[w, \sigma(\varphi_1)] = [v, \sigma(\varphi_2)]$. Thus, we have shown that ρ is well-defined. Note that if $[s, \varphi] \in \mathcal{G}_u(\mathcal{S})$ and $w, v \in \mathcal{W}_s$, then $[w, \sigma(\varphi)] = [v, \sigma(\varphi)]$ (apply the above result in the case that $t = s$ and $\varphi_2 = \varphi_1$).

Next, we show that ρ is a groupoid homomorphism. Let $\iota : E(\mathcal{W}) \to E(\mathcal{S})$ be the isomorphism given by (W1), and $\sigma : \widehat{E(\mathcal{S})} \to \widehat{E(\mathcal{W})}$ be the induced homeomorphism. First, we check that ρ preserves composability. Suppose $([s, \varphi], [t, \gamma]) \in \mathcal{G}_u(\mathcal{S})^{(2)}$. We wish to show that $(\rho([s, \varphi]), \rho([t, \gamma])) \in \mathcal{G}_u(\mathcal{W})^{(2)}$, and so let $w \in \mathcal{W}_s$ and $v \in \mathcal{W}_t$. Notice first that $\theta_t(\gamma) = \varphi$. Then, we have

$$\begin{aligned} \theta_v(\sigma(\gamma)) &= \sigma(\theta_t(\gamma)) \\ &= \sigma(\varphi), \end{aligned}$$

and so $d([w, \sigma(\varphi)]) = r([v, \sigma(\gamma)])$ as desired.

Now, we check that ρ is compatible with the groupoid multiplication. Let

$$([s, \varphi], [t, \gamma]) \in \mathcal{G}_u(\mathcal{S})^{(2)},$$

and take $w \in \mathcal{W}_s$ and $v \in \mathcal{W}_t$. Then, using part (i) of Lemma 4.2.27, we have

$$\begin{aligned}\rho([s,\varphi])\rho([t,\gamma]) &= [w,\sigma(\varphi)][v,\sigma(\gamma)]\\ &= [wv,\sigma(\gamma)]\\ &= \rho([st,\gamma])\\ &= \rho([s,\varphi][t,\gamma]).\end{aligned}$$

Next, we ensure ρ preserves inverses. Let $[s,\varphi] \in \mathcal{G}_u(\mathcal{S})$, and supose $w \in \mathcal{W}_s$. Then,

$$\begin{aligned}\rho([s,\varphi])^{-1} &= [w,\sigma(\varphi)]^{-1}\\ &= [w^*,\theta_w(\sigma(\varphi))]\\ &= [w^*,\sigma(\theta_s(\varphi))]\\ &= \rho([s^*,\theta_s(\varphi)])\\ &= \rho([s,\varphi]^{-1}).\end{aligned}$$

Hence, $\rho([s,\varphi])^{-1} = \rho([s,\varphi]^{-1})$ as desired, and so we have shown that ρ is a groupoid homomorphism.

Next, we show that ρ is bijective. Surjectiveness follows immediately from (W2), as if $[w,\sigma(\varphi)] \in \mathcal{G}_u(\mathcal{W})$ then there exists $s \in \mathcal{S}$ such that $\varphi \in D_{s^*s}$ and

$$\sigma(\theta_s(\varphi)) = \theta_w(\sigma(\varphi)).$$

Hence, $w \in \mathcal{W}_s$ and so $\rho([s,\varphi]) = [w,\sigma(\varphi)]$. In other words, the sets $\mathcal{W}_s$ cover $\mathcal{G}_u(\mathcal{W})$.

To see injectivity holds, suppose that $\rho([s,\varphi]) = \rho([t,\gamma])$ for some $s,t \in \mathcal{S}$ with $\varphi \in D_{s^*s}$ and $\gamma \in D_{t^*t}$. Let $w \in \mathcal{W}_s$ and $v \in \mathcal{W}_t$. Then we have $[w,\sigma(\varphi)] = [v,\sigma(\gamma)]$. This is only true if $\sigma(\varphi) = \sigma(\gamma)$ in which case $\varphi = \gamma$, since σ is a homeomorphism. Now, the above equality implies that there exists some idempotent $f \in E(\mathcal{W})$ such that $\sigma(\varphi) \in D_f$ and $wf = vf$. We remark that $wf = vf$ holds regardless of what elements of $\mathcal{W}_s$ and $\mathcal{W}_t$ we choose. In particular, we have $\varphi(\iota(f)) = 1$ and, by property (W3), there exists $e \in E(\mathcal{S})$ such that $\varphi(e) = 1$ and $se = te$, and so we have $[s, \varphi] = [t, \gamma]$.

It remains to show that ρ is a homeomorphism. This amounts to showing that both ρ and ρ^{-1} are continuous. In the first case, let $U \subseteq \mathcal{G}_u(\mathcal{W})$ be open. Let $[s, \varphi] \in \mathcal{G}_u(\mathcal{S})$ and $w \in \mathcal{W}_s$. The inverse mapping ρ^{-1} is given by $\rho^{-1}([w, \sigma(\varphi)]) = [s, \varphi]$, and is well-defined by virtue of ρ being bijective. Suppose that $U = \Theta(w, V)$ is a basic open set, where $s \in \mathcal{W}$ and $V \subseteq D_{w^*w}$ is open. Now,

$$\rho^{-1}(U) = \{[s, \varphi] \in \mathcal{G}_u(\mathcal{S}) : \sigma(\varphi) \in V\} = \{[s, \varphi] \in \mathcal{G}_u(\mathcal{S}) : \varphi \in \sigma^{-1}(V)\}.$$

Since σ is continuous, $\sigma^{-1}(V)$ is open in $\widehat{E(\mathcal{S})}$ and therefore open in $\mathcal{G}_u(\mathcal{S})$ ($\mathcal{G}_u(\mathcal{S})$ is étale and so has open unit space), and we have $\rho^{-1}(U) = \Theta(s, \sigma^{-1}(V))$, which is open.

Conversely, let $U \subseteq \mathcal{G}_u(\mathcal{S})$ be open. As before, suppose $U = \Theta(s, V)$ for some $s \in \mathcal{S}$ and open $V \subseteq D_{s^*s}$. Let $w \in \mathcal{W}_s$. Then,

$$\rho(U) = \{[w, \sigma(\varphi)] \in \mathcal{G}_u(\mathcal{S}) : \varphi \in V\} = \{[w, \sigma(\varphi)] \in \mathcal{G}_u(\mathcal{S}) : \sigma(\varphi) \in \sigma(V)\}.$$

Since σ is an open map, $\sigma(V)$ is open and so $\rho(U) = \Theta(w, \sigma(V))$, which is a basic open set of $\mathcal{G}_u(\mathcal{W})$. Hence, ρ and ρ^{-1} are continuous and so ρ is a homeomorphism. □

The converse seems to only hold if we begin with the assumption that the idempotents are isomorphic. The issue arises when attempting to construct an isomorphism between $E(\mathcal{S})$ and $E(\mathcal{W})$ via the collection of compact open bisections on the character spaces - this method fails to work when the collection $\mathcal{D}(E(\mathcal{S}))$ doesn't form a basis for the topology on the semicharacter space.

Proposition 4.2.30. Let $\mathcal{S}$, $\mathcal{W}$ be inverse semigroups with $\mathcal{W} \subseteq \mathcal{S}$ and $E(\mathcal{S}) \cong E(\mathcal{W})$. If $\mathcal{G}_u(\mathcal{S}) \cong \mathcal{G}_u(\mathcal{W})$, then $\mathcal{W}$ is wide in $\mathcal{S}$.

Proof. Since we assume that $E(\mathcal{S}) \cong E(\mathcal{W})$, it remains to show that (W2) and (W3) hold.

Suppose $\iota : E(\mathcal{W}) \to E(\mathcal{S})$ is an isomorphism, and let $\sigma : E(\mathcal{S}) \to E(\mathcal{W})$ be the homeomorphism given by Equation 4.2. Without loss of generality, we show that $\Gamma_{\mathcal{S}}(\widehat{E(\mathcal{S})}) \subseteq \sigma^{-1}\Gamma_{\mathcal{W}}(\widehat{E(\mathcal{W})})\sigma$. So, let $\theta_s \in \Gamma_{\mathcal{S}}(\widehat{E(\mathcal{S})})$, and $\varphi \in D_{s^*s}$. Suppose $w \in \mathcal{W}_s$. Then,

$$r(\rho([s, \varphi])) = r([w, \sigma(\varphi)]),$$

since ρ is a groupoid homomorphism. This implies that

$$\sigma(\theta_s(\varphi)) = \theta_w(\sigma(\varphi)),$$

which then gives us

$$\theta_s(\varphi) = \varphi^{-1}(\theta_w(\sigma(\varphi))).$$

Hence, $\Gamma_{\mathcal{S}}(\widehat{E(\mathcal{S})}) \subseteq \sigma^{-1}\Gamma_{\mathcal{W}}(\widehat{E(\mathcal{W})})\sigma$. A similar argument shows that $\Gamma_{\mathcal{W}}(\widehat{E(\mathcal{W})}) \subseteq \sigma^{-1}\Gamma_{\mathcal{S}}(\widehat{E(\mathcal{S})})\sigma$ i.e. $\Gamma_{\mathcal{S}}(\widehat{E(\mathcal{S})}) \supseteq \sigma^{-1}\Gamma_{\mathcal{W}}(\widehat{E(\mathcal{W})})\sigma$. Hence, we have equality as desired, and so (W2) is satisfied.

It remains to show that $\mathcal{W}$ satisfies (W3). Let $s, t \in \mathcal{S}$ and $F_\varphi \in \mathcal{F}(E(\mathcal{S}))$, as well as $w \in \mathcal{W}_s$ and $v \in \mathcal{W}_t$. Suppose there exists $e \in F$ such that $se = te$, but there is no idempotent $f \in E(\mathcal{W})$ such that $\iota(f) \in F_\varphi$ and $wf = vf$. That is, for every idempotent $\iota(h) \in F_\varphi$, one has $wh \neq vh$, and so one cannot have $[w, \sigma(\varphi)] = [v, \sigma(\varphi)]$. Since ρ is a homeomorphism, this gives us that

$$\rho^{-1}([w, \sigma(\varphi)]) = [s, \varphi] \neq [t, \varphi] = \rho^{-1}([v, \sigma(\varphi)]).$$

But this contradicts our assumption that there exists $e \in F$ satisfying $se = te$. Hence, there must exist such an $f \in E(\mathcal{W})$. An identical argument shows that the converse implication holds. $\square$

4.3 The Groupoid of Ultragerms

Throughout this section, let $\mathcal{S}$ be a Boolean inverse semigroup, such that $E(\mathcal{S})$ is a generalized Boolean algebra. Recall that $\mathrm{Spec}(E(\mathcal{S}))$ denotes the set of Boolean homomorphisms from $E(\mathcal{S})$ to the 2-element Boolean algebra $\{0,1\}$. This is commonly referred to as the *Stone space of* $E(\mathcal{S})$ (see, for instance, [Law23, p. 8-9]). As with the spectrum, we topologize the Stone space of $E(\mathcal{S})$ with the inherited product topology from the space $\{0,1\}^{E(\mathcal{S})}$, where $\{0,1\}$ is discrete.

4.3.1 The Topology on $\operatorname{Spec}(E(\mathcal{S}))$

We show below that the Boolean structure on $\operatorname{Spec}(E(\mathcal{S}))$ allows its topology to have as a basis the collection $\mathcal{D}(E(\mathcal{S}))$, contrary to $\widehat{E(\mathcal{S})}$ and $\widehat{E_0(\mathcal{S})}$, which require the finer collection $\operatorname{Patch}(E(\mathcal{S}))$.

The following result is mentioned in [Ste23, p. 6], but we provide a proof.

Proposition 4.3.1. The collection $\mathcal{D}(E(\mathcal{S})) = \{D_e : e \in E(\mathcal{S})\}$ is a basis for the inherited product topology on $\operatorname{Spec}(E(\mathcal{S}))$.

Proof. We use [Mun03, Lemma 13.2] by checking that the collection above satisfies the condition whereby for every open U in $\operatorname{Spec}(E(\mathcal{S}))$ and every $\varphi \in U$, then there exists e such that $\varphi \in D_e \subseteq U$. It is useful to note that by swapping the order of quantifiers, this is equivalent to requiring that for every $\varphi \in \operatorname{Spec}(E(\mathcal{S}))$ and open U containing φ, there exists e such that $\varphi \in D_e \subseteq U$. Suppose that U is an open subset of $\operatorname{Spec}(E(\mathcal{S}))$ and $F_\varphi \in U$ is a non-trivial filter. Since $\operatorname{Spec}(E(\mathcal{S}))$ is topologized as a subspace of $\widehat{E(\mathcal{S})}$, which is generated by the patch topology, we may assume that

$$U = U_{e;X} \cap \operatorname{Spec}(E(\mathcal{S})) = D_e \cap \bigcap_{x \in X} D_x^c \cap \operatorname{Spec}(E(\mathcal{S}))$$

for some $e \in E(\mathcal{S})$ and $X \subseteq_{\text{fin}} E(\mathcal{S})$. We define

$$f := e \setminus (e \wedge \bigvee_{x \in X} x),$$

and now claim that D_f satisfies our condition. First, we check that $\varphi \in D_f$. Since φ is a Boolean homomorphism to the two-element Boolean algebra, we have

$$\begin{aligned}\varphi(f) &= \varphi(e \setminus (e \wedge \bigvee_{x \in X} x)) \\ &= \varphi(e) \setminus (\varphi(e) \wedge \bigvee_{x \in X} \varphi(x)) \\ &= 1 \setminus (1 \wedge 0) \\ &= 1 \setminus 0 \\ &= 1.\end{aligned}$$

It remains to show that $D_f \subseteq U_{e;X} \cap \mathrm{Spec}(E(\mathcal{S}))$. So, let $\psi \in D_f$. Since $f = e \setminus (e \wedge \bigwedge_X x) \leq e$, and $\psi(f) = 1$, it follows that $\psi(e) = 1$, since filters are upward-closed. Now toward a contradiction, suppose that $\psi(x') = 1$ for some $x' \in X$. Then,

$$\psi(fex') = \psi(f)\psi(e)\psi(x') = 1.$$

However, $f \perp (e \wedge \bigvee_{x \in X} x)$ meaning $f(e \wedge \bigvee_{x \in X} x) = 0$. The collection X is pointwise compatible - notice that any two idempotents e, f are trivially compatible since the set of idempotents is closed under taking inverses and multiplication. This is true also of the set $\{e \wedge x : x \in X\}$. Thus, using an identity from [Weh17, Proposition 3.1.9], this means that

$$0 = f(e \wedge \bigvee_X x) = f(\bigvee_X e \wedge x) = \bigvee_X (f \wedge e \wedge x),$$

which in particular implies $f \wedge e \wedge x' = 0$. But $\psi(fex') = 1$, meaning $\psi(0) = 1$, which contradicts ψ being a Boolean homomorphism. □

Similarly to how we defined an action of $\mathcal{S}$ on $\widehat{E(\mathcal{S})}$, we can define an action of $\mathcal{S}$ on $\mathrm{Spec}(E(\mathcal{S}))$ simply by restriction, since $\mathrm{Spec}(E(\mathcal{S})) \subseteq \widehat{E(\mathcal{S})}$ and $\mathrm{Spec}(E(\mathcal{S}))$ is an invariant subset of $\widehat{E(\mathcal{S})}$ (see [ACaH+22, Section 3.2]). We denote by $\mathcal{G}_\infty(\mathcal{S})$ the groupoid of germs of this action, and following [ACaH+22], call it the *groupoid of ultragerms*. Then $\mathcal{G}_\infty(\mathcal{S})$ is an étale, ample groupoid with a basis given by the collection $\{\Theta(s, U) : s \in$

$\mathcal{S}, U \subseteq_\circ D_{s^*s}\}$. Notice that the groupoid $\mathcal{G}_\infty(\mathcal{S})$ embeds as a subgroupoid into $\mathcal{G}_u(\mathcal{S})$.

Proposition 4.3.2. Let $\mathcal{S}$ and $\mathcal{W}$ be Boolean inverse semigroups with ultragerm groupoids $\mathcal{G}_\infty(\mathcal{S})$ and $\mathcal{G}_\infty(\mathcal{W})$. If $\rho : \mathcal{G}_\infty(\mathcal{S}) \to \mathcal{G}_\infty(\mathcal{W})$ is a homeomorphism, then for every $e \in E(\mathcal{S})$, we have $\rho(D_e) = D_f$ for some $f \in E(\mathcal{W})$.

Proof. Fix $e \in E(\mathcal{S})$. Since the collection $\{D_e : e \in E(\mathcal{S})\}$ forms a basis for $\mathrm{Spec}(E(\mathcal{S}))$, and ρ and its inverse are continuous, we can assume that $\rho(D_e) = \bigcup_{i \in I} D_i$ for some index set $I \subseteq E(\mathcal{W})$. But ρ is a homeomorphism, and in particular maps compact sets to compact sets. Since D_e is compact, so is $\rho(D_e)$, and so there exists a finite $J \subseteq I$ such that $\rho(D_e) = \bigcup_{i \in J} D_i$. Defining

$$j \coloneqq \bigvee_{i \in J} i,$$

we have $j \in E(\mathcal{W})$, since $E(\mathcal{W})$ admits finite meets. In this case, $\rho(D_e) = D_j$, and so $\rho(D_e)$ is of the form D_i for some idempotent $i \in E(\mathcal{W})$. □

4.3.2 Generalized Stone Duality

Recall that the generalized Boolean algebra of compact open sets on $\mathrm{Spec}(E(\mathcal{S}))$ is exactly the collection $\mathcal{D}(E(\mathcal{S})) = \{D_e : e \in E(\mathcal{S})\}$. The meet and join operations are given by set intersection and union, respectively.

Lemma 4.3.3. *Let $\mathcal{S}$ be a Boolean inverse semigroup. The mapping*

$$\tau_{\mathcal{S}} : E(\mathcal{S}) \to \mathcal{D}(E(\mathcal{S}))), \quad e \mapsto D_e \tag{4.4}$$

is an isomorphism of generalized Boolean algebras.

Proof. We begin by showing that τ is bijective. It is certainly surjective, as for any $D_e \in \mathcal{D}(E(\mathcal{S}))$ we have $e \mapsto D_e$. To see that it is injective, let $e, f \in E(\mathcal{S})$ and suppose $D_e = D_f$. Then $e \in F$ if and only if $f \in F$ for every filter F on $E(\mathcal{S})$. In particular, since $e \in e^\uparrow$, we have $f \in e^\uparrow$ implying

$f \in e^{\uparrow}$, which means $f \geq e$. Similarly, we have $e \geq f$. Thus, $e = f$, and so τ is injective.

It remains to show that τ is a homomorphism of generalized Boolean algebra. If $e, f \in E(\mathcal{S})$, then

$$\tau(e \wedge f) = \tau(ef) = D_{ef} = D_e \cap D_f = D_e \wedge D_f.$$

Now, we note that if $\varphi \in \mathrm{Spec}(E(\mathcal{S}))$ such that $\varphi(e \vee f) = 1$, then since φ is a Boolean homomorphism, we have that either $\varphi(e) = 1$ or $\varphi(f) = 1$, and so $D_{e \vee f} = D_e \cup D_f$. In this case,

$$\tau(e \vee f) = D_{e \vee f} = D_e \cup D_f = D_e \vee D_f.$$

Lastly, we check that τ preserves the 0 element. In $E(\mathcal{S})$ this is the zero idempotent 0. The elements of D_0 are those characters φ such that $\varphi(0) = 1$, which implies D_0 is empty, as no such characters exist. Hence D_0 is the 0 element of $\mathcal{D}(E(\mathcal{S}))$. □

Lemma 4.3.4. *Let $\rho : \mathcal{G}_\infty(\mathcal{S}) \to \mathcal{G}_\infty(\mathcal{W})$ be a homeomorphism, with homeomorphism between unit spaces $\rho' : \mathrm{Spec}(E(\mathcal{S})) \to \mathrm{Spec}\, E(\mathcal{W})$. Then, the induced map*

$$\hat{\rho} : \mathcal{D}(E(\mathcal{S}))) \to \mathcal{D}(E(\mathcal{W}), \quad D_e \mapsto \rho'(D_e) \tag{4.5}$$

is an isomorphism of generalized Boolean algebras.

Proof. The isomorphism $\hat{\rho}$ is well defined by Proposition 4.3.2. We first check that $\hat{\rho}$ is bijective. If $f' \in E(\mathcal{W})$, then $D_{f'} \in \mathcal{D}(E(\mathcal{S}))$, and $\rho^{-1}(D_{f'}) = D_f$ for some $f \in E(\mathcal{S})$. Then, $\hat{\rho}(D_f) = D_{f'}$. Now, suppose $\hat{\rho}(D_e) = \hat{\rho}(D_f) = D_u$. Then $\rho(D_e) = \rho(D_f) = D_u$, and so $\rho^{-1}(D_u) = D_e = D_f$. Lastly, we check that $\hat{\rho}$ is a (generalized) Boolean homomorphism. The zero element of $\mathcal{D}(E(\mathcal{S}))$ is the empty set, and clearly $\rho(\emptyset) = \emptyset$. If $D_e, D_f \in \mathcal{D}(E(\mathcal{S}))$, we claim that $\hat{\rho}(D_e) \wedge \hat{\rho}(D_f) = \hat{\rho}(D_e \wedge D_f)$. Suppose that $\hat{\rho}(D_e) = D_{e'}$ and $\hat{\rho}(D_f) = D_{f'}$. We first claim that

$$\hat{\rho}(D_{ef}) = D_{(ef)'} = D_{e'f'} = \hat{\rho}(D_e)\hat{\rho}(D_f).$$

We know that $\rho(\varphi) \in D_{(ef)'}$ if and only if $\varphi \in D_{ef}$, which is true if and only if $\varphi \in D_e \cap D_f$. In this case,

$$\varphi \in \rho(D_e) \cap \rho(D_f) = D_{e'} \cap D_{f'} = D_{e'f'}.$$

Hence, $D_{(ef)'} = D_{e'f'}$. We can now see that

$$\hat{\rho}(D_e \wedge D_f) = \hat{\rho}(D_{ef}) = D_{(ef)'} = D_{e'f'} = D_{e'} \cap D_{f'} = \hat{\rho}(D_e) \wedge \hat{\rho}(D_f).$$

It remains to check that $\hat{\rho}$ respects the relative complement and join operations. Suppose $D_e \subseteq D_f$, such that $D_f \setminus D_e$ exists. We have that $\rho(\varphi) \in \rho(D_e \setminus D_f)$ if and only if $\varphi \in D_e$ and $\varphi \notin D_f$, in which case $\rho(\varphi) \in \rho(D_e) \setminus \rho(D_f)$. Hence, $\hat{\rho}(D_f \setminus D_e) = \hat{\rho}(D_f) \setminus \hat{\rho}(D_e)$. Finally, for $D_e, D_f \in \mathcal{D}(E(\mathcal{S}))$, we have $\rho(\varphi) \in D_{e'} \cup D_{f'}$ if and only if $\varphi \in D_e$ or $\varphi \in D_f$, if and only if $\varphi \in D_e \cup D_f$, which is equivalent to having $\rho(\varphi) \in \rho(D_e \cup D_f)$. Hence, $\hat{\rho}(D_e \vee D_f) = \hat{\rho}(D_e) \vee \hat{\rho}(D_f)$. □

Lemma 4.3.5. *Let $\rho : \mathcal{G}_\infty(\mathcal{S}) \to \mathcal{G}_\infty(\mathcal{W})$ be a groupoid homeomorphism with restriction $\rho' : \mathrm{Spec}(E(\mathcal{S})) \to \mathrm{Spec}\, E(\mathcal{W})$, induced isomorphism $\hat{\rho} : \mathcal{D}(E(\mathcal{S})) \to \mathcal{D}(E(\mathcal{W}))$ given by Equation 4.5, and isomorphisms $\tau_\mathcal{S}, \tau_\mathcal{W}$ as defined by Equation 4.4. Then, the map*

$$\iota : E(\mathcal{S}) \to E(\mathcal{W}), \quad e \mapsto \tau_\mathcal{W}^{-1}(\hat{\rho}(\tau_\mathcal{S}(e)))$$

is an isomorphism of generalized Boolean algebras.

Proof. We have that ι is a composition of generalized Boolean algebra isomorphisms, and so is itself a generalized Boolean algebra isomorphism. □

Recall that the sets $\Theta(s, D_{s^*s})$ are compact open bisections (see Proposition 3.2.8). Below, we show that we don't need every single bisection $\Theta(s, D_e)$.

Lemma 4.3.6. *Let $\mathcal{S}$ be a Boolean inverse semigroup. Then $\mathrm{Bis}_c(\mathcal{G}_\infty(\mathcal{S})) = \{\Theta(s, D_{s^*s}) : s \in \mathcal{S}\}$.*

Proof. We wish to show that the collections $\{\Theta(s, D_e) : s \in \mathcal{S}, e \in E(\mathcal{S})\}$ and $\{\Theta(s, D_{s^*s}) : s \in \mathcal{S}\}$ coincide. One can see that the latter is trivially contained in the former, so we show the reverse inclusion holds. Let $\Theta(s, D_e)$ be a compact open bisection - we claim that

$$\Theta(s, D_e) = \Theta(se, D_{(se)^*(se)}).$$

If $[s, \varphi] \in \Theta(s, D_e)$ such that $\varphi \in D_e$, then we have $\varphi \in D_e \cap D_{s^*s} = D_{es^*s} = D_{(se)^*(se)}$. Furthermore, notice that $see = se$, and so since $\varphi \in D_e$, we have $[s, \varphi] = [se, \varphi]$ and so $\Theta(s, D_e) \subseteq \Theta(se, D_{(se)^*(se)})$. Similarly, if $[se, \varphi] \in \Theta(se, D_{(se)^*(se)})$, then $[se, \varphi] = [s, \varphi]$ and $\varphi \in D_{s^*s}$, therefore $\Theta(se, D_{(se)^*(se)}) \subseteq \Theta(s, D_{s^*s})$. Hence, we have shown every compact open bisection of $\mathcal{G}_\infty(\mathcal{S})$ is equal to one of the form $\Theta(s, D_{s^*s})$ for some $s \in \mathcal{S}$. □

We now extend $\tau_{\mathcal{S}}$ to a map $\psi_{\mathcal{S}} : \mathcal{S} \to \mathrm{Bis}_c(\mathcal{G}_\infty(\mathcal{S}))$ given by

$$s \mapsto \Theta(s, D_{s^*s}) = \{[s, \varphi] : \varphi \in D_{s^*s}\}. \tag{4.6}$$

By the above lemma, we know that the image of $\psi_{\mathcal{S}}$ indeed coincides with $\mathrm{Bis}_c(\mathcal{G}_\infty(\mathcal{S}))$. We now show that it is an inverse semigroup isomorphism. The following isomorphism is mentioned in [Ste23, 3.3], and a proof via groupoids of filters is provided in [LL13] (in particular, see Proposition 2.9 and Lemma 3.11). We give an alternate proof.

Lemma 4.3.7. *Let $\mathcal{S}$ be a Boolean inverse semigroup. Then the map given by*

$$\psi_{\mathcal{S}} : \mathcal{S} \to \mathrm{Bis}_c(\mathcal{G}_\infty(\mathcal{S})), \quad s \mapsto \Theta(s, D_{s^*s})$$

is an isomorphism of inverse semigroups.

Proof. We first check that $\psi_{\mathcal{S}}$ is a homomorphism. Let $s, t \in \mathcal{S}$. Using

[Exe08, Proposition 4.5], we first note that $D_{(st)^*(st)} = \theta_{t^*}(D_{s^*s} \cap D_{t^*t})$. Then,

$$\begin{aligned}\psi_{\mathcal{S}}(st) &= \Theta(st, D_{(st)^*(st)}) \\ &= \{[st, \varphi] : \varphi \in D_{(st)^*(st)}\} \\ &= \{[st, \varphi] : \phi = \theta_t(\varphi) \in D_{s^*s} \cap D_{tt^*}\} \\ &= \{[s, \phi][t, \varphi] : \phi = \theta_t(\varphi) \in D_{s^*s} \cap D_{tt^*}\} \\ &= \{[s, \phi][t, \varphi] : \phi \in D_{s^*s}, \varphi \in D_{t^*t} \\ &= \Theta(s, D_{s^*s})\Theta(t, D_{t^*t}) \\ &= \psi_{\mathcal{S}}(s)\psi_{\mathcal{S}}(t).\end{aligned}$$

Hence, $\psi_{\mathcal{S}}$ preserves multiplication. To see that it preserves inverses, we have

$$\begin{aligned}\psi_{\mathcal{S}}(s^*) &= \Theta(s^*, D_{ss^*}) \\ &= \{[s^*, \varphi] : \varphi \in D_{ss^*}\} \\ &= \{[s^*, \theta_s(\phi)] : \phi \in D_{s^*s}\} \\ &= \Theta(s, D_{s^*s})^* \\ &= \psi_{\mathcal{S}}(s)^*.\end{aligned}$$

It remains to show that $\psi_{\mathcal{S}}$ is bijective. It is straightforward to check that it is surjective, since the sets $\Theta(s, D_{s^*s})$ give all compact open bisections by Lemma 4.3.6.

Now, for injectivity, suppose $\psi_{\mathcal{S}}(s) = \psi_{\mathcal{S}}(t)$. That is, $\Theta(s, D_{s^*s}) = \Theta(t, D_{t^*t})$. Then $[s, \varphi] \in \Theta(s, D_{s^*s})$ if and only if there exists $[t, \phi] \in \Theta(t, D_{t^*t})$ such that $[s, \varphi] = [t, \phi]$. In particular, this implies $\varphi = \phi$, thus $D_{s^*s} = D_{t^*t}$. Note that since the map $e \mapsto D_e$ is an isomorphism (Proposition 4.3.3), we must have $s^*s = t^*t$. We claim that $s = t$.

For each $\varphi \in D_{s^*s}$, there exists $e_\varphi \in E(\mathcal{S})$ such that $\varphi \in D_{e_\varphi}$ and $se_\varphi = te_\varphi$. Consider the union

$$W := \bigcup_{\varphi \in D_{s^*s}} D_{e_\varphi}.$$

Certainly W is an open cover of D_{s^*s}, and since D_{s^*s} is compact, we can pass to a finite collection $I \subseteq \mathrm{Spec}(E(\mathcal{S}))$ such that

$$W = \bigcup_{\varphi \in I} D_{e_\varphi}.$$

But since $\mathcal{S}$ is Boolean, it admits finite joins of idempotents, and so $\bigvee_{\varphi\in I} e_\varphi \in E(\mathcal{S})$. Since τ is a generalized Boolean algebra homomorphism (see Equation 4.4), we have

$$\bigcup_{\varphi \in I} D_{e_\varphi} = D_{\bigvee_{\varphi\in I} e_\varphi}.$$

We now define

$$W' := D_{s^*s} \cap W = D_{(s^*s)(\bigvee_{\varphi\in I} e_\varphi)}.$$

Notice that W' is a finite union and intersection of compact clopen sets, so is itself compact and clopen. In this case, as $D_{s^*s} \subseteq W'$, we have

$$W' = D_{(s^*s)(\bigvee_{\varphi\in I} e_\varphi)} = D_{s^*s},$$

and so $(s^*s)(\bigvee_{\varphi\in I} e_\varphi) = s^*s$ i.e. $s^*s \leq \bigvee_{\varphi\in I} e_\varphi$. Note that this also holds for t^*t. Now, applying Proposition 3.1.8, as well as the definition of germ equivalence, we have

$$\begin{aligned} s = ss^*s = ss^*s(\bigvee_{\varphi\in I} e_\varphi) = s(\bigvee_{\varphi\in I} e_\varphi) = \bigvee_{\varphi\in I} se_\varphi \\ = \bigvee_{\varphi\in I} te_\varphi = t(\bigvee_{\varphi\in I} e_\varphi) = tt^*t(\bigvee_{\varphi\in I} e_\varphi) = tt^*t = t. \end{aligned}$$

Hence, $s = t$ as desired.

We have shown that if $\mathcal{S}$ is a Boolean inverse semigroup, then the map $\psi_{\mathcal{S}} : \mathcal{S} \to \mathrm{Bis}_c(\mathcal{G}_\infty(\mathcal{S}))$ is an isomorphism of inverse semigroups. □

Lemma 4.3.8. *Let $\mathcal{S}$ and $\mathcal{W}$ be Boolean inverse semigroups, such that there exists a homeomorphism $\rho : \mathcal{G}_\infty(\mathcal{S}) \to \mathcal{G}_\infty(\mathcal{W})$. Then the induced map given by*

$$\hat{\rho} : \mathrm{Bis}_c(\mathcal{G}_\infty(\mathcal{S})) \to \mathrm{Bis}_c(\mathcal{G}_\infty(\mathcal{W})), \quad \Theta(s, D_{s^*s}) \mapsto \rho(\Theta(s, D_{s^*s}))$$

is an isomorphism of inverse semigroups.

Proof. We have seen that the collection of open bisections on an étale groupoid form an inverse semigroup. Furthermore, the collection of compact open bisections on an ample groupoid form an inverse semigroup, often called the ample inverse semigroup [Ren21, Proposition A9]. Hence, it remains to show that $\hat{\rho}$ is an inverse semigroup isomorphism.

To see that $\hat{\rho}$ is surjective, notice that we can write any compact open bisection of $\mathcal{G}_\infty(\mathcal{W})$ as $\Theta(s, D_{s^*s})$ for some $s \in \mathcal{W}$. Then, $\rho^{-1}(\Theta(s, D_{s^*s})$ is a compact open bisection of $\mathcal{G}_\infty(\mathcal{S})$. Furthermore, injectivity of $\hat{\rho}$ follows from the injectivity of ρ and the well-definedness of ρ^{-1}.

Let $s, t \in \mathcal{S}$, so that $\Theta(s, D_{s^*s})$ and $\Theta(t, D_{t^*t})$ are compact open bisections. We have

$$\begin{aligned}\Theta(s, D_{s^*s})\Theta(t, D_{t^*t}) &= \{[s, \varphi][t, \phi] : s, t \in \mathcal{S}, \varphi \in D_{s^*s}, \phi \in D_{t^*t}, \varphi = \theta_t(\phi)\} \\ &= \{[st, \phi] : st \in \mathcal{S}, \phi \in D_{(st)^*(st)}\} \\ &= \Theta(st, D_{(st)^*(st)}).\end{aligned}$$

Hence, $\hat{\rho}$ respects multiplication of compact open bisections. Now, if $s \in \mathcal{S}$, we have

$$\begin{aligned}\Theta(s, D_{s^*s})^{-1} &= \{[s, \varphi]^{-1} : s \in \mathcal{S}, \varphi \in D_{s^*s}\} \\ &= \{[s^*, \theta_s(\varphi)] : s \in \mathcal{S}, \varphi \in D_{s^*s}\} \\ &= \{[s^*, \phi] : s \in \mathcal{S}, \phi \in D_{ss^*}\} \\ &= \Theta(s^*, D_{ss^*}).\end{aligned}$$

Thus, $\hat{\rho}$ respects inversion of compact open bisections. Therefore, $\hat{\rho}$ is an inverse semigroup homomorphism, and we have shown it is an isomorphism. □

One might want to show a similar result to 4.2.22 for Boolean characters, but it turns out that such a functor doesn't necessarily map Boolean characters to other Boolean characters. That is, if $\iota : E \to E'$ is a morphism of generalized Boolean algebras, then the image of $F(\iota)$ as defined by

$$F(\iota)(\varphi) = \varphi \circ \iota$$

doesn't coincide with $\mathrm{Spec}(E')$. In this case, we must take a more direct approach.

Lemma 4.3.9. *Let E, E' be generalized Boolean algebras, and let $\iota : E \to E'$ be an isomorphism. Define*

$$\sigma_U : \mathrm{Spec}(E') \to \mathrm{Spec}(E), \quad \varphi \mapsto \varphi \circ \iota. \tag{4.7}$$

Then σ_U is a homeomorphism.

Proof. We first check that σ_U indeed maps ultrafilters to ultrafilters. We use the criteria introduced by Exel in [Exe08, Lemma 12.3], which states that U is an ultrafilter on E if and only if U contains every element y such that $y \Cap x$ for all $x \in U$. So, let U be an ultrafilter on E' and consider $\sigma_U(U)$. By the proof of 4.2.22, we know that $\sigma_U(U)$ is at least a filter, so it remains to check it is an ultrafilter. Let $y \in E$, and suppose $y \Cap x$ for all $x \in \sigma_U(U)$. That is, $y \wedge x \neq 0$ for all $x \in \sigma_U(U)$. Then, using the fact that ι is injective and preserves meets and the 0, we have

$$\iota(y) \wedge \iota(x) = \iota(y \wedge x) \neq \iota(0) = 0.$$

That is, $\iota(y) \Cap \iota(x)$. This is true for all $x \in \sigma_U(U)$, and so holds for all $\iota(x) \in U$. Thus, since U is an ultrafilter, we have $\iota(y) \in U$, and hence $y \in \sigma_U(U)$ as desired.

Next, we check that σ_U is bijective. Let $\varphi, \psi \in \mathrm{Spec}(E')$, and suppose $\sigma(\varphi) = \sigma(\psi)$. That is, $\varphi(\iota(e)) = \psi(\iota(e))$ for all $e \in E$. But ι is surjective, and so $\varphi(f) = \psi(f)$ for all $f \in E'$. Hence, $\varphi = \psi$ and σ_U is injective. To see it is surjective, let $\varphi \in \mathrm{Spec}(E)$. Define a character ϕ on E' by

$$\phi(e) = \varphi(\iota^{-1}(e)).$$

Since ι is an isomorphism, so is ι^{-1}, and so ϕ is a Boolean character. Furthermore, we have

$$\sigma_U(\phi)(e) = \phi(\iota(e)) = \varphi(\iota(\iota^{-1}(e))) = \varphi(e).$$

Hence, σ_U is surjective and thus bijective.

We can define σ_U^{-1} by

$$\sigma_U^{-1}(\varphi)(e) = \varphi(\iota^{-1}(e)),$$

which is well-defined in virtue of ι being an isomorphism. A similar argument to that in the proof of Lemma 4.2.22 shows that σ_U and σ_U^{-1} are continuous. Hence, σ_U is a homeomorphism. □

Theorem 4.3.10. Let $\mathcal{S}$ and $\mathcal{W}$ be Boolean inverse semigroups. Then $\mathcal{S} \cong \mathcal{W}$ if and only if $\mathcal{G}_\infty(\mathcal{S}) \cong \mathcal{G}_\infty(\mathcal{W})$.

Proof. Suppose that $\mathcal{S}$ and $\mathcal{W}$ are both Boolean inverse semigroups, such that there exists a homeomorphism $\rho : \mathcal{G}_\infty(\mathcal{S}) \to \mathcal{G}_\infty(\mathcal{W})$ with induced map $\hat{\rho} : \mathrm{Bis}_c(\mathcal{G}_\infty(\mathcal{S})) \to \mathrm{Bis}_c(\mathcal{G}_\infty(\mathcal{W}))$. By Lemma 4.3.8, $\hat{\rho}$ is an isomorphism of inverse semigroups. Consider the following composition of maps, where $\psi_\mathcal{S}$ and $\psi_\mathcal{W}$ are defined as in Equation 4.6.

$$\psi_\mathcal{W}^{-1} \circ \hat{\rho} \circ \psi_\mathcal{S} : \mathcal{S} \to \mathcal{W}.$$

Then $\psi_\mathcal{W}^{-1} \circ \hat{\rho} \circ \psi_\mathcal{S}$ is a composition of inverse semigroup isomorphisms, and thus is itself an inverse semigroup isomorphism between $\mathcal{S}$ and $\mathcal{W}$.

We show that the converse implication holds. If $\iota : \mathcal{S} \to \mathcal{W}$ is an isomorphism, and $\sigma : \mathrm{Spec}(E(\mathcal{W})) \to \mathrm{Spec}(E(\mathcal{S}))$ is the induced map, then by Lemma 4.3.9 we know that σ is a homeomorphism between unit spaces. Define a map ρ_U as follows.

$$\rho_U : \mathcal{G}_\infty(\mathcal{S}) \to \mathcal{G}_\infty(\mathcal{W}), \quad [s, \varphi] \mapsto [\iota(s), \sigma^{-1}(\varphi)].$$

An identical argument to that of 4.2.25 shows that ρ_U is a homeomorphism of groupoids. □

Chapter 5

Applications and Future Investigation

In the following sections, we investigate how universal groupoids and groupoids of ultragerms can be characterized in a variety of contexts. For instance, we discuss the relationship between the corresponding groupoids of a group with a zero adjoined, as well as a link between the finite symmetric inverse semigroup and the finite symmetric group (in particular, their representations as one-object categories). We then discuss some further questions and possible future routes of investigation.

5.1 Applications

First, we briefly discuss C^*-algebras of inverse semigroups and groupoids, and how our notion of wideness relates to Paterson's results linking the C^*-algebras of an inverse semigroup with that of its universal groupoid.

5.1.1 The C^*-Algebra of the Universal Groupoid

We apply our results to those obtained by Paterson in [Pat99, Theorem 4.4.1, 4.4.2], which show that both the full and reduced C^*-algebras of an inverse semigroup are isomorphic to that of its universal groupoid. We direct the reader to his book for a detailed construction of these algebras, and proof of this result.

Let H be a Hilbert space, and consider $\mathcal{B}(H)$, the collection of bounded operators on H. We say that $a \in \mathcal{B}(H)$ is a *projection* if $a^2 = a = a^*$, and we say a is a *partial isometry* if a^*a is a projection ([Put19, Definition 1.1.3]).

It is well-documented that every inverse semigroup can be represented as the collection of partial isometries on a Hilbert space (see, for instance, [Pat99, Proposition 2.1.4]). If $\mathcal{S}$ is an inverse semigroup, one can define a *representation of $\mathcal{S}$ on H* to be a $*$-homomorphism $\pi : \mathcal{S} \to \mathcal{B}(H)$. It follows that if $e \in E(\mathcal{S})$ then $\pi(e)$ is a projection, and so $\pi(s)$ is a partial isometry for each $s \in \mathcal{S}$. One can use such representations to construct both the full and reduced C^*-algebras of $\mathcal{S}$ (see [Pat99, Section 2.1]).

Similarly, if $\mathcal{G}$ is an arbitrary étale groupoid, then one can construct its full and reduced C^*-algebras $C^*(\mathcal{G})$, by taking the completions of $C_c(\mathcal{G})$ with respect to particular norms. This process is well-described in, for instance, [Sim18, Chapter 3] and [Exe08, Section 3].

Corollary 5.1.1. Let $\mathcal{S}, \mathcal{W}$ be Boolean inverse semigroups such that $\mathcal{W} \subseteq \mathcal{S}$. If $\mathcal{W}$ is wide in $\mathcal{S}$, then $C^*(\mathcal{S}) \cong C^*(\mathcal{W})$.

Proof. Since $\mathcal{W}$ is wide in $\mathcal{S}$, we have $\mathcal{G}_u(\mathcal{S}) \cong \mathcal{G}_u(\mathcal{S})$. By [Pat99, Theorem 4.4.1], we have $C^*(\mathcal{S}) \cong C^*(\mathcal{G}_u(\mathcal{S}))$ and $C^*(\mathcal{W}) \cong C^*(\mathcal{G}_u(\mathcal{W}))$. Hence, $C^*(\mathcal{S}) \cong C^*(\mathcal{W})$. □

Remark 5.1.2. In [Ste10], for some inverse semigroup $\mathcal{S}$ and commutative ring with unit K, Steinberg introduces the semigroup algebra $K\mathcal{S}$, and describes this as a convolution algebra of functions on the universal groupoid $\mathcal{G}_u(\mathcal{S})$ of $\mathcal{S}$. We suggest further enquiry into the relationship between the Steinberg algebra of the universal groupoid of an inverse semigroup $\mathcal{S}$, and the semigroup algebra $K\mathcal{S}$.

5.1.2 Groups with an Adjoined Idempotent

Consider the inverse semigroup $\mathcal{S}$ obtained by taking a group with identity element 1. Since the idempotents consist of a single element, one might consider $\mathcal{S}$ to be a Boolean inverse semigroup where the idempotents are the trivial Boolean algebra. However, this case is uninteresting, and so we instead consider a slight extension by adjoining an additional idempotent.

If G is a group, we can adjoin a 0 element to the group to obtain an inverse semigroup $\mathcal{S} = G \cup \{0\}$, such that 0 satisfies $0g = g0 = 0$ for all $g \in G$, and $0^{-1} = 0$. Then, $E(\mathcal{S}) \cong \{0, 1\}$, where $\{0, 1\}$ is the two-element Boolean algebra. In this case, $\mathcal{S}$ admits a non-empty collection of filters, consisting of the filters $\{\{0\}, \{0, 1\}\}$.

Proposition 5.1.3. Let G be a group, and $\mathcal{S} = G \cup \{0\}$ be an inverse semigroup with 0 adjoined. Then $\mathcal{G}_u(\mathcal{S}) \cong G \cup \{0\}$, and $\mathcal{G}_0(\mathcal{S}) \cong \mathcal{G}_\infty(\mathcal{S}) \cong G$.

Proof. Since G is a group, the set of idempotents $E(\mathcal{S})$ consists of a single element, say e, which is the identity of the group, along with the adjoined 0. We remark that this makes G into a Boolean inverse semigroup with idempotents isomorphism to the two-element Boolean algebra. This implies there exists only one non-zero character $\varphi \in \widehat{E(\mathcal{S})}$ given by $\varphi(e) = 1$ and $\varphi(0) = 0$, which is also a Boolean character. This means that $\widehat{E(\mathcal{S})} = \mathrm{Spec}(E(\mathcal{S})) = \{\varphi\}$, implying that both $\mathcal{G}_u(\mathcal{S})$ and $\mathcal{G}_\infty(\mathcal{S})$ are a group.

It remains to show that this group is identical to G. Note that $\mathcal{G}_u(\mathcal{S})$ consists of elements $\{(s, \varphi) : s \in \mathcal{S}\}$ modulo the germ equivalence relation. However, in this context, no pairs are equivalent. If $[s, \varphi] = [t, \varphi]$, then we must have $sf = tf$ for some idempotent f, but f must either be the identity e of G, or 0. However, $\varphi \notin D_0 = \emptyset$, and so we must have $f = e$, implying $s = t$. That is, each germ in the groupoid is a singleton. Therefore $\mathcal{G}_u(\mathcal{S}) = \{[s, \varphi] : s \in \mathcal{S}\}$. Define a map $\pi : \mathcal{G}_u(\mathcal{S}) \to G$ given by $[s, \varphi] \mapsto s$. The previous statement shows that π is injective, and surjectivity follows from the fact that $s^*s = 0$ and so $\varphi \in D_{s^*s}$ for every $s \in G$.

Lastly, we show that π is a semigroup homomorphism. Let $[s, \varphi], [t, \varphi] \in \mathcal{G}_u(\mathcal{S})$. We know that the product of these elements is $[st, \varphi]$, which is mapped by π to $st \in G$. Hence, $\pi(st) = \pi(s)\pi(t)$. Furthermore, $[s, \varphi]^{-1} = [s^*, \varphi] \mapsto s^* = s^{-1}$. Thus, π is a semigroup homomorphism. □

Proposition 5.1.4. Let $\mathcal{I}_n$ be the symmetric inverse semigroup on n elements. Then $\mathcal{G}_\infty(\mathcal{I}_n)$ is the unique contractible groupoid on n objects, up to isomorphism.

Proof. We have that the ultrafilters on $E(\mathcal{I}_n)$ consist of the principal ultrafilters generated by primitive elements, which correspond to singletons of $E(\mathcal{I}_n)$. That is, $\mathrm{Spec}(E(\mathcal{I}_n)) \cong \{1, 2, \ldots, n\}$, and so $\mathcal{G}_\infty(\mathcal{I}_n)$ has objects $\{1^\uparrow, 2^\uparrow, \ldots, n^\uparrow\}$. Furthermore, if $x, y \in \{1, 2, \ldots, n\}$, then the partial bijection $x \mapsto y$ takes the ultrafilter $x^\uparrow$ to the ultrafilter $y^\uparrow$, and so the action of $\mathcal{I}_n$ on $\mathrm{Spec}(E(\mathcal{I}_n))$ is transitive. Hence, $\mathrm{Hom}(x^\uparrow, y^\uparrow) = \{*\}$ for every $x, y \in \{1, 2, \ldots, n\}$.

We see that $\mathcal{G}_\infty(\mathcal{I}_n)$ is contractible, since every hom-set consists of a single isomorphism. Furthermore, if $\mathcal{G}'$ is any contractible groupoid with objects $\{\bar{1}, \bar{2}, \ldots, \bar{n}\}$, we can define a map $\mathcal{G}' \to \mathcal{G}_\infty(\mathcal{I}_n)$ that maps $\bar{1}$ to $1^\uparrow$, and an arrow $\bar{x} \to \bar{y}$ to the unique arrow $x^\uparrow \to y^\uparrow$. □

Remark 5.1.5. We note that the unique contractible groupoid with n objects is isomorphic to the full equivalence relation on a set X with n elements - that is, $\mathcal{G}_\infty(\mathcal{I}_n) \cong X \times X$, where $|X| = n$.

Proposition 5.1.6. Every subgroup of $\mathbb{Z}$, viewed as an inverse semigroup, is wide in $\mathbb{Z}$.

Proof. Let $H \subseteq \mathbb{Z}$ be a subgroup of $\mathbb{Z}$. Since $\mathbb{Z}$ is cyclic, every subgroup is of the form $n\mathbb{Z}$ for some $n \in \mathbb{N}$. That is, $H = m\mathbb{Z}$ for some $m \in \mathbb{N}$. But then $\mathbb{Z} \cong H$ via the isomorphism $x \mapsto mx$. Hence, $\mathcal{G}_u(\mathbb{Z}) \cong \mathcal{G}_u(H)$. We know that the idempotents are isomorphic (being a singleton), and so by Proposition 4.2.30 H is wide in $\mathbb{Z}$. □

5.2 Future Investigation

5.2.1 Minimally Wide Sub-Semigroups

We have seen that an inverse semigroup $\mathcal{S}$ may give rise to sub-semigroups with the same universal groupoid as $\mathcal{S}$. One natural question one may ask is whether a *minimal* wide sub-semigroup exists.

Given an inverse semigroup $\mathcal{S}$, let $\mathrm{Wide}(\mathcal{S})$ denote the collection of wide sub-semigroups of $\mathcal{S}$. Then $\mathrm{Wide}(\mathcal{S})$ is a partially ordered set, where the order is given by inclusion of sub-semigroups. This order is partial since, as we have seen, an inverse semigroup may admit pairs of wide sub-semigroups, such that neither contains the other. Furthermore, it is not true in general that unions or intersections of wide sub-semigroups are themselves wide. However, the relation "is a wide sub-semigroup of" is transitive, due to the transitivity of groupoid homeomorphisms.

In general, there exists no "least" wide sub-semigroup, in the sense that it is contained in every other wide sub-semigroup. However, a totally ordered chain $\mathcal{C}$ of wide sub-semigroups may admit a greatest lower bound i.e. a sub-semigroup $\mathcal{W}$ such that $\mathcal{W} \subseteq \mathcal{T}$ for all $\mathcal{T} \in \mathcal{C}$.

In particular, let $\{W_i\}$ be a chain of wide sub-semigroups of $\mathcal{S}$, such that $W_i \supseteq W_{i+1}$ for all i. Then $\inf(\{W_i\}) = \bigcap_i W_i$, but in general, $\inf(\{W_i\})$ is not wide. One expects that it is (W1) that fails, as the process of passing to the intersection of wide sub-semigroups loses idempotents.

Example 5.2.1. We consider again the inverse semigroup $\mathbb{Z} \cup \{e\}$. The collection of wide inverse subsemigroups of $\mathbb{Z} \cup \{e\}$ is a partially ordered set

$$\{\mathbb{Z} \cup \{e\}, 2\mathbb{Z} \cup \{e\}, 3\mathbb{Z} \cup \{e\}, \ldots\}.$$

Note that here,

$$E(\mathbb{Z} \cup \{e\}) = E(2\mathbb{Z} \cup \{e\}) = \ldots \cong \{0, 1\}.$$

This is order-isomorphic to the partially ordered set of natural numbers ordered by the "divides" relation (whereby $a \leq b$ if and only if a divides b).

This collection of inverse semigroups has no minimum element, but has infimum the two-element inverse semigroup consisting of the idempotents $\{0, 1\}$. In particular,

$$E(\inf\{n\mathbb{Z} \cup \{e\}\}) \cong E(\mathbb{Z} \cup \{e\}) \cong \{0, 1\}.$$

Remark 5.2.2. We propose further investigation into the existence and characterization of minimal wide sub-semigroups, and conjecture that such a characterization may be obtained by a suitable application of Zorn's lemma to this problem. However, detailed discussion of this falls outside the scope of this thesis.

5.2.2 Path Groupoids

In his 2002 paper, Paterson describes how one can generate an inverse semigroup from an arbitrary directed graph Σ, with vertex set V and edge set E. He then applies the universal groupoid construction, obtaining an ample groupoid that coincides with what is called the path groupoid. Elements of this groupoid are triples of the form $(\alpha\gamma, l(\alpha) - l(\beta), \beta\gamma)$ where α, β are finite paths in Σ, γ is a finite or infinite path, and $l(\alpha)$ denotes the length of the path α. The unit space of this groupoid consists of the triples of the form $(\alpha\gamma, 0, \alpha\gamma)$, and so can be identified with the collection of finite and infinite paths.

There is an invariant subset of the unit space consisting of only the paths that are either infinite, or finite but terminate at an infinite emitter - that is, a vertex v with $d^{-1}(v) = \infty$. The reduction of the groupoid to this subset is another commonly studied groupoid, called the boundary path groupoid of Σ [Rig, Section 2]. Paterson proves that the C^*-algebras of Σ and the boundary path groupoid of Σ are isomorphic.

Take a directed graph Σ, and let z be a distinguished zero element. The *graph inverse semigroup* of G is the inverse semigroup $\mathcal{S}_G$ generated by $V \cup E \cup E^* \cup \{z\}$ following the relations below.

(i) For all $s \in \mathcal{S}$, we have $0s = s0 = 0$.

(ii) For all $e \in E$, we have $d(e)e = e$ and $er(e) = e$. Furthermore, $e^*d(e) = e^*$ and $r(e)e^* = e^*$.

(iii) If $a, b \in V \cup E \cup E^*$ and $r(a) \neq d(b)$, then $ab = z$.

(iv) If $e, f \in E$ and $e \neq f$ then $e^*f = z$.

Notice that for an arbitrary element of $\mathcal{S}_G$, many of the products reduce to vertices or go to z. Hence, without loss of generality, we can write an element of $\mathcal{S}_G$ as

$$\alpha_1\alpha_2 \ldots \alpha_n {\beta_k}^* {\beta_{k-1}}^* \ldots {\beta_1}^*$$

where each $\alpha_i, \beta_i \in V \cup E$.

Paterson models the inverse semigroup $\mathcal{S}$ by the collection T of pairs (α, β) where α, β are finite paths with matching ranges. One can identify $\alpha\beta^* \in \mathcal{S}$ with $(\alpha, \beta) \in T$. We can then construct the universal groupoid of $\mathcal{S} \cong T$. We have that $E(T) = \{(\alpha, \alpha) : \alpha \in Y\}$, and so we can identify the idempotents of T with the finite paths in G. The natural partial order on T is such that $\alpha \geq \beta$ if and only if α is an initial segment of β - that is, if there exists $\gamma \in Y$ such that $\beta = \alpha\gamma$. In this way, filters on $E(T)$ correspond to (possibly infinite) paths.

Question 5.2.3. Let Σ be a directed graph, with graph inverse semigroup $\mathcal{S}$. Does $\mathcal{S}$ admit any wide sub-semigroups?

That is, are there any sub-semigroups $\mathcal{W} \subseteq \mathcal{S}$ such that $\mathcal{G}_u(\mathcal{W}) \cong \mathcal{G}_u(\mathcal{S})$? Furthermore, could such a sub-semigroup be characterized in terms of the underlying directed graph?

One candidate is the sub-semigroup $\mathcal{S}^n$ of $\mathcal{S}$ given by the collection $\{(\alpha\gamma, \beta\gamma) : l(\alpha) = l(\beta) = n\} \cup \{z\}$. Paterson asserts that $E(\mathcal{S}^n) = E(\mathcal{S})$, which certainly implies that $\mathcal{S}^n$ satisfies the first condition of wideness. The universal groupoid of $\mathcal{S}_n$ can be identified with the collection $\{(\alpha\gamma, 0, \beta\gamma) : l(\alpha) = l(\beta) = n\}$, which is a subgroupoid of $\mathcal{G}_u(\mathcal{S})$. However, as before, in-depth discussion of this falls outside our current scope.

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