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Countable and Dependent Choice Equivalences in Topological Groupoids

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Abstract

Working in ZF, we begin by proving that every complete pseudometric space is Baire if and only if every complete metric space is Baire. We then show some results that investigate the relationship between topological groupoids being topologically principle and effective, and use these to add topological groupoid-related equivalences to the Axiom of Countable Choice, as well as the Axiom of Dependent Choice.

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Chapter 1

Introduction

Algebra and topology are often considered at first to be very different flavours of mathematics - one very structured, and the other more fluid. Despite this, due to a topological space being something that we can establish on any set, at the intersection of these two fields lies algebraic structures that are also topological spaces. The study of topological groups, for instance, is commonplace. In this thesis, we are interested particularly in topological groupoids.

A groupoid has many equivalent definitions, including one that is category-theoretical, but algebraically a groupoid is a weakening of a group (every group is a groupoid), such that the binary operation on the group may only be a partial function. Intuitively, a groupoid can be seen as what we call a "unit space" of objects (elements of the form $\gamma\gamma^{-1}$, where γ^{-1} denotes the inverse of an element with respect to our binary operation) and arrows (non-unit elements of the groupoid) between objects, symbolizing composition of elements. Groupoids also usually come along with range and source maps, which map each element of the groupoid (each arrow) to the object that they coming from/going to. A groupoid can also be a topological space, but in order to be considered a topological groupoid, it must satisfy a few properties; namely, that it is locally compact and all the relevant groupoid maps are continuous (range, source, inverse and composition). A further subclass of topological groupoids, called étale groupoids, will be of particular interest. This is because of a number of nice properties they possess; for instance, if G is an étale groupoid, the unit space $G^{(0)}$ is open in G, and the range map r is a local homeomorphism from G to G.

In their 2000 article, Herrlich and Keremedis [4] discuss equivalences between different forms of the axiom of choice and a particular property that topological spaces may possess - that of being a Baire space (in which the countable intersection of open dense sets is itself dense). There exist various formulations of the Baire category theorem, which gives sufficient conditions for a topological space to be Baire. We relate these to properties of topological groupoids, introduced by Brown and Clark [2], such as those of being (weakly) effective and topologically principle. Briefly, a groupoid is weakly effective if $\mathrm{Iso}(G) - G^{(0)}$ has empty interior, where $\mathrm{Iso}(G)$ denotes the isotropy subgroupoid of G (the subgroupoid formed from all the elements $x \in G$ such that $r(x) = s(x) = x$) and expand the equivalences found in theorems 0.15 and 0.16 [2] by adding new hypotheses that relate both to pseudo-metric spaces and metric spaces being Baire, and topological groupoids being effective and/or topologically principle.

Our paper establishes the following main results, both of which are extensions of theorems 0.15 and 0.16 of Herrlich and Keremedis' paper.

Theorem (5.1)**.** The following are equivalent:

(1) let G be an étale groupoid that has a countable cover of open bisections, such that $G^{(0)}$ is a second-countable complete metric space. If G is weakly effective, it is topologically principle.

(2) the Axiom of Countable Choice.

Theorem (5.8)**.** The following are equivalent:

(1) let G be an étale groupoid with a countable cover of open bisections, such that $G^{(0)}$ is a complete metric space. If G is weakly effective, it is topologically principle.

(2) the Axiom of Dependent Choice.

In section 2 we discuss some preliminaries, focusing on topological spaces, properties of topological spaces, and set theory, including ZFC. Working in ZF henceforth, in section 3 we then prove that every complete pseudometric space is Baire if and only if every complete metric space is Baire. In section 4, we introduce the idea of topological groupoids and the properties of being effective and topologically principle, and discuss the relationship between these properties and look at a number of examples to demonstrate this. Finally, in section 5 we prove the main results as are mentioned above.

Chapter 2

Preliminaries

2.1 Topology

We begin by defining some preliminary topological ideas. All of the following definitions can be found in [10].

Let X be a set. A *topology* on X is a collection τ of subsets of X such that the following conditions hold;

(a) arbitrary unions of sets belonging to τ also belong to τ;

(b) finite intersections of members of τ are also members of τ;

(c) both X and $\emptyset$ are members of τ.

The pair (X, τ) is then a *topological space*. For any set $U \subseteq X$, if $U \in \tau$ then U is said to be *open*. A set $V \subseteq X$ is *closed* if $U = X \setminus V$ is open.

If U is a set, then the *interior* of U (denoted $\text{Int}(U)$) is the union of all open sets contained in U. The closure of U (denoted $\bar{U}$) is then the intersection of all closed sets which contain U. Every interior is an open set, and every closure is a closed set. We can then define the *boundary* of U as $\partial U = \bar{U} \setminus \text{Int}(U)$. Let X be a topological space. Then a subset A of X is *dense* if $\bar{A} = X$, and *nowhere dense* if $\text{Int}(\bar{A}) = \emptyset$;

A topological space X is said to be *Hausdorff* if, for every pair of points $x, y \in X$, there exist open neighbourhoods U and V of x and y, respectively, which are disjoint.

Remark. The definition of a topological neighbourhood differs between many authors, but here a *neighbourhood of* x is any open set U which contains the point x.

For any topological space X, a collection $\mathcal{C}$ of open subsets of X is said to be an *open cover* of X if

$$X = \bigcup_{C \in \mathcal{C}} C.$$

A subcollection $\hat{\mathcal{C}}$ of $\mathcal{C}$ is an *open subcover* of X if $\hat{\mathcal{C}} \subseteq \mathcal{C}$ and this subcollection itself covers X. We say X is *compact* if every open cover of X admits a finite subcover. Furthermore, X is *locally compact* if, for every $x \in X$, there exists a neighbourhood U of x such that U is compact.

A *basis* of a topological space X is a collection of sets $\mathcal{B}$ such that

(a) $\mathcal{B}$ covers X (in other words, for every $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$);

(b) if $x \in B_1 \cap B_2$ for two basis elements B_1, B_2, then there exists a third basis element B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$.

A result of this is that every open set U can be described as the union of basis elements.

There are a number of specific topologies on sets that we can define, and that prove to be useful. Let (X, τ_X) be a topological space, and take $Y \subseteq X$. Then the *subspace topology* induced in Y by X is defined as follows; a set $U \subseteq Y$ is open in Y if and only if $U = V \cap Y$ for some set $V \in \tau_X$. It is then easy to show that Y is indeed a topological space.

Let X be a topological space along with an equivalence relation $\sim$. Let Y be the set of equivalence classes of X under $\sim$.

Similarly, let $(X\tau_X), (Y, \tau_Y)$ be topological spaces, and let $X \times Y$ be the Cartesian product of the two underlying sets. Then the set $\{U \times V : U \in \tau_X, V \in \tau_Y\}$ forms a basis for the *product topology* on $X \times Y$.

A *metric space* is a pair (X, d) where X is a set, and d is a binary function $d : X \times X \to \mathbb{R}$ called a *metric* satisfying the following properties:

(a) $d(x, y) \geq 0$ for all $x, y \in X$;

(b) $d(x, y) = 0 \iff x = y$ for all $x, y \in X$;

(c) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

If (X, d) satisfies

$$x = y \implies d(x, y) = 0 \text{ for all } x, y \in X$$

in place of (b), then (X, d) is called a *pseudometric space*.

It is worth noting that the topology on a pseudometric space is generated by the basis of open balls $B(x; \varepsilon)$ for $x \in X$ and $\varepsilon > 0$, similarly to a metric space [10, p. 23].

Let X be a topological space. Then X is a *Baire space* if either X is empty, or satisfies the following equivalent conditions:

(a) X is not the countable union of a sequence $\{A_n\}$ of nowhere dense sets in X;

(b) the intersection of any sequence $\{D_n\}$ of dense, open subsets of X is non-empty [4, Def. 0.12];

(c) given any countable collection $\{A_n\}$ of closed sets of X, each of which has empty interior in X, their union $\bigcup A_n$ also has empty interior in X [7, p. 295].

2.2 Set Theory

Axiomatic set theory plays an important role in the theorems we prove. In particular, a theory called Zermelo-Fraenkel set theory (ZFC) forms the foundation for most of modern mathematics - in this case, theory refers to a collection of axioms, as opposed to the more colloquial meaning. It will be helpful if we briefly look at ZFC - more specifically, the axiom of choice as well as a few variations. The following provides an informal overview of the axioms [1].

(1) **Extensionality** - two sets x and y are equal iff they have the same elements.

(2) **Empty Set** - there exists a set x such that for every y, $y \notin x$.

(3) **Pairing** - for all sets x, y, there exists a set with x and y as elements (the unordered pair).

(4) **Union** - if x and y are sets, then there exists a set $x \cup y$ whose elements are those that are either in x or y (or both).

(5) **Comprehension** - if x is a set and φ is a first-order sentence, then there exists a subset of x of elements that satisfy φ.

(6) **Replacement** - if x is a set and f is a function, then the image of x under f is a set.

(7) **Power set** - if x is a set, then there exists a set whose elements are exactly the subsets of x.

(8) **Infinity** - there exists an infinite inductive set - that is, there exists a set x such that $\emptyset \in x$, and whenever $a \in x$, we have $a \cup \{a\} \in x$ (this relies on the definition of Von Neumann ordinals).

(9) **Foundation** - if x is a set, then there exists an element $y \in x$ such that $x \cap y = \emptyset$.

(10) **Choice** - if $\mathcal{C}$ is a non-empty collection of non-empty sets, then there exists a function $f : \mathcal{C} \to \bigcup_{C \in \mathcal{C}} C$ such that $f(C) \in C$ for every element C of the collection.

Most of the axioms of ZFC are widely accepted, and the axiom of choice is the most controversial - assuming choice (ZFC) implies many important results which require this axiom to be proven. However, there also exist multiple weakenings of choice. The axiom of countable choice, for instance, we will use here and it states that for each countable sequence of sets, there exists a choice function which maps each set to a member of itself. Note that this is indeed weaker than choice, as choice can apply to uncountable families of sets.

A slightly different formulation is the axiom of dependent choice, which states that if φ is a total binary relation on a set R, then there exists a sequence $(x_n) \subseteq R$ such that for every n, $x_n \varphi x_{n+1}$. That is, we can find a sequence in R such that each pair of consecutive elements are related by φ. As it turns out, dependent choice implies countable choice, but the reverse implication is false.

A final formulation of choice we will consider is dependent multiple choice, which states that if φ is a total binary relation on a set R, then there exists a sequence (X_n) of non-empty, finite subsets of R such that for all n and all $x \in X_n$, there exists some $y \in X_{n+1}$ such that $x \varphi y$. That is, for each set in the sequence and every element in the set, we can find an element in the following set such that the relation between them holds. [4]

Since we are working to establish equivalences between properties of topological spaces and various weakened versions of the axiom of choice, it should be noted that throughout this thesis (unless otherwise stated) we are working in ZF.

Chapter 3

Preservation of the Baire property of metric spaces

In this section we establish the following theorem. It is known that this theorem is true, by the equivalences given by Herrlich and Keremedis [4, Thm. 0.15, 0.16], but we provide a direct topological proof.

Theorem 3.1. Every complete metric space is Baire if and only if every complete pseudometric space is Baire.

Before proceeding with the proof of the above theorem, we will first consider some intermediate results.

Let (X, d) be a pseudometric space. We define an equivalence relation $\sim$ on X defined by $x \sim y$ iff $d(x, y) = 0$. That this is indeed an equivalence relation is easy to verify. Let X' be the set of equivalence classes of $\sim$, with a metric $d_\sim$ defined by $d_\sim([x], [y]) = d(x, y)$ Producing a metric space from a pseudometric space via this equivalence relation is called metric identification [5]. We will verify that this is indeed a well-defined metric - that is, the distance between two equivalence classes of points is the same regardless of which specific representative points are chosen.

Lemma 3.2. *The function $d_\sim([x], [y])$ on the quotient space $X/\sim$ is a well-defined metric.*

Proof. Suppose $[x_1] = [x_2]$ and $[y_1] = [y_2]$. We wish to show that the metric is well-defined i.e. $d_\sim([x_1], [y_1]) = d_\sim([x_2], [y_2])$. Since $[x_1] = [x_2]$, $d(x_1, x_2) = 0$. Similarly, $d(y_1, y_2) = 0$. Then,

$$\begin{aligned} d_\sim([x_1], [y_1]) &= d(x_1, y_1) \\ &\leq d(x_1, x_2) + d(x_2, y_2) + d(y_2, y_1) \\ &= 0 + d(x_2, y_2) + 0 \\ &= d([x_2], [y_2]). \end{aligned}$$

Thus, the metric $d_\sim([x], [y])$ is well-defined on $X/\sim$. It remains to show that this function does indeed define a metric.

Since d is a pseudometric, we have $d(x, y) \geq 0$ for all $x, y \in X$. But then

$$d_\sim([x], [y]) = d(x, y) \geq 0.$$

Now, suppose $d_\sim([x],[y]) = 0$. We wish to show that $[x] = [y]$. Without loss of generality, let $a \in [x]$ i.e. $d(a,x) = 0$. Then,

$$d(a,y) \leq d(a,x) + d(x,y) = d(a,x) + d_\sim([x],[y]) = 0 + 0 = 0.$$

Hence $d(a,y) = 0$, and so $a \in [y]$. This gives us $[x] \subseteq [y]$, and a similar process shows $[y] \subseteq [x]$. Therefore, $[x] = [y]$. In the other direction, suppose $[x] = [y]$, and we wish to show $d_\sim([x],[y]) = 0$. Since $x \in [x] = [y]$ and $y \in [y] = [x]$, we have $d(x,y) = 0$ which implies $d_\sim([x],[y]) = 0$.

Lastly, we check the triangle inequality. Let $[x],[y],[z] \in X/\sim$. Then,

$$\begin{aligned} 0 \leq d_\sim([x],[z]) &= d(x,z) \\ &\leq d(x,y) + d(y,z) \\ &= d_\sim([x],[y]) + d_\sim([y],[z]), \end{aligned}$$

so the triangle inequality holds. Thus, we have shown that $d_\sim$ indeed defines a metric. □

Lemma 3.3. *If (X,d) is a complete pseudometric space, then $(X/\sim, d_\sim)$ is also complete.*

Proof. Let $\{[x]_n\}$ be a Cauchy sequence in $(X/\sim, d_\sim)$. We wish to show that $\{[x]_n\}$ converges in the metric space. Since (X,d) is complete, the sequence $\{x_n\}$ converges in X if it is Cauchy. We check that $\{x_n\}$ is indeed Cauchy. Fix $\varepsilon > 0$. Since $\{[x]_n\}$ is Cauchy, there exists $N \in \mathbb{N}$ such that

$$n,m \geq N \implies d_\sim([x]_n,[x]_m) \leq \varepsilon.$$

Taking any $n,m \geq N$, observe that

$$0 \leq d(x_n,x_m) = d_\sim([x]_n,[x]_m) < \varepsilon.$$

Hence, $\{x_n\}$ is a Cauchy sequence also and there exists $x \in X'$ such that $x_n \to x$. Since, for all $x,y \in X$, we have

$$d(x,y) = d_\sim([x],[y]),$$

we can see that

$$d_\sim([x]_n,[x]) = d(x_n,x) \to 0 \text{ as } n \to \infty.$$

Hence $\{[x]_n\}$ converges to $[x]$, so $(X/\sim, d_\sim)$ is complete. □

Thus, using metric identification on a complete pseudometric space produces a complete metric space. We now wish to see whether using metric identification preserves the property of being Baire (to be precise, the property of not-being-Baire!)

Henceforth we denote $X/\sim$ as X' for convenience. Consider the function $\phi : X \to X'$, defined by $\phi(x) = [x]$.

Lemma 3.4. *The function ϕ is a quotient mapping from X to X'.*

Proof. Notice that ϕ is surjective (for every $[x] \in X'$, we have $\phi(x) = [x]$).

We first prove ϕ is an open map i.e. maps open sets to open sets. To see this, let $A \subset X$ be an open set, and let $\phi(a) \in \phi(A)$. Since A is open, there exists $\varepsilon > 0$ such that $B_X(a;\varepsilon) \subset A$. We wish to show that

$$\phi(B_X(a;\varepsilon)) = B_{X'}(\phi(a);\varepsilon).$$

Take $y \in B_X(a;\varepsilon)$ and consider $\phi(y) \in \phi(B_X(a;\varepsilon))$. This means that $d(a,y) < \varepsilon$. But,

$$d(a,y) = d_{\sim}(\phi(a),\phi(y)) < \varepsilon,$$

and so $\phi(y) \in B_{X'}(\phi(a);\varepsilon)$. This then gives

$$\phi(B_X(a;\varepsilon)) \subseteq B_{X'}(\phi(a);\varepsilon).$$

Conversely, take $\phi(z) \in B_{X'}(\phi(a);\varepsilon)$. Note that such a point is guaranteed since ϕ is surjective. This means that

$$d_{\sim}(\phi(a),\phi(z)) < \varepsilon.$$

Then,

$$d_{\sim}(\phi(a),\phi(z)) = d(a,z) < \varepsilon,$$

meaning $z \in B_X(a;\varepsilon)$. Then $\phi(z) \in \phi(B_X(a;\varepsilon))$ giving us

$$B_{X'}(\phi(a);\varepsilon) \subseteq \phi(B_X(a;\varepsilon)).$$

Now, since these two open balls are equal, we have

$$B_X(a;\varepsilon) \subset A \implies \phi(B_X(a;\varepsilon)) = B_{X'}(\phi(a);\varepsilon) \subset \phi(A).$$

Hence, $\phi(A)$ is open.

For ϕ to be continuous, for every basis element $V = B_{X'}([x];\varepsilon)$ of X', we need to have $\phi^{-1}(V)$ open. By the definition of the function ϕ, $V = B_{X'}(\phi(x);\varepsilon)$. But, by the previous part of the proof, we know that $B_{X'}(\phi(x);\varepsilon) = \phi(B_X(x;\varepsilon)$. Hence,

$$\phi^{-1}(B_{X'}(\phi(x);\varepsilon)) = \phi^{-1}(\phi(B_X(x;\varepsilon))) = B_X(x;\varepsilon).$$

Since $B_X(x;\varepsilon)$ is clearly open in X, ϕ is continuous. □

Corollary 3.5. X' is a quotient space of X.

Lemma 3.6. *Let X, X' and ϕ be the spaces and mapping defined as above. Suppose A is a dense subset of X. Then $\phi(A)$ is dense in X'.*

Proof. Suppose A is dense in X. We first observe that ϕ is a surjective mapping (for every $[y] \in X'$, there exists $y \in X$ such that $\phi(y) = [y]$). Take any point $[y] \in X'$. Then $[y] = \phi(y)$ for some $y \in X$. For any $\varepsilon > 0$, we wish to show that $B_{X'}[y];\varepsilon)$ contains some element of $\phi(A)$ i.e. $\phi(A)$ is dense. Since ϕ is continuous, $\phi^{-1}(B_X([y];\varepsilon)) = B_{X'}(y;\varepsilon)$ is open (technically the inverse image of $B_X([y];\varepsilon)$ is a union of balls of points in X, but these collapse down into a single ball since the distance between their midpoints are all 0). Since A is dense in X, this means that $B_X(y;\varepsilon)$ contains some member $a \in A$. Hence, since ϕ preserves distances between points, $\phi(B'_X(y;\varepsilon) = B_{X'}([y];\varepsilon)$ contains the point $\phi(a) \in \phi(A)$. Therefore, $\phi(A)$ is dense in X'. □

Lemma 3.7. *If X is not Baire, then X' is not Baire.*

Proof. If X is not Baire, then there exists a sequence $\{D_n\}$ of open dense subsets of X such that $\bigcap D_n = \emptyset$. We wish to construct a sequence $\{B_n\} \subset X'$ of open, dense subsets of X such that $\bigcap B_n = \emptyset$. Since each set D_i is open and dense in X, $\phi(D_i)$ is open and dense in X'. Hence, we consider the sequence $\{\phi(D_n)\} \subset X'$, each of which is open and dense. Suppose toward a contradiction that $\bigcap \phi(D_n) \neq \emptyset$. Hence, there exists some element $[x] \in \phi(D_i)$

for each $i \in \mathbb{N}$. Furthermore, since ϕ is a surjective mapping, there exists $x \in X$ such that $\phi(x) = [x]$. We can see that it must be that $x \in D_i$ for each i (there may be other points associated with the equivalence class $[x]$ not in D_i, but at least one must be in D_i in order for $[x]$ to be in $\phi(D_i)$).

Hence, $x \in D_n$ for each D_n i.e. $x \in \bigcap D_n$. But this contradicts our assumption that X is Baire. It then must be the case that $\bigcap \phi(D_n) = \emptyset$. So, X' is not Baire. □

We now have the sufficient tools to prove Theorem 3.1, which stated that every complete metric space is Baire if and only if every complete pseudometric space is Baire.

Proof of Theorem 3.1. Suppose that every complete metric space is Baire, but that not every complete pseudometric space is Baire. Then, there exists a complete pseudometric space (X, d) that is not Baire. By Lemmas 3.2 and 3.3, we can use the function $\phi : X \to X'$ to transform X into a metric space, preserving completeness. Then, by Lemma 3.7, we know that the mapping ϕ preserves the property of not being a Baire space. Hence, our metric space is complete but not Baire. Since this contradicts our initial assumption, it must be the case that every complete pseudometric space is Baire. The reverse implication follows trivially since every metric space is a pseudometric space. □

By extension, we can say that if every complete metric space is Baire, then every second countable complete pseudometric space is Baire, however the reverse implication is not true (this would require the second countability of the complete metric space).

We can go further and establish an equivalence between these properties in second countable metric and pseudometric spaces. In doing so, we expand Theorem 0.15 [4] with the addition of a further equivalence.

Corollary 3.8. The following are equivalent:

(i) every totally bounded complete pseudometric space is Baire,

(ii) every second countable complete pseudometric space is Baire,

(iii) every second countable complete metric space is Baire,

(iv) the Axiom of Countable Choice.

Proof. $(ii) \implies (iii)$: Clearly if every s.c. complete pseudometric space is Baire, then since every metric space is a pseudometric space, it is trivial that every s.c. metric space is Baire.

$(iii) \implies (ii)$: Every complete metric space is Baire implies every complete pseudometric space is Baire follows from Theorem 3.1. All that remains to prove is that the property of second-countability is preserved when forming a metric space from a pseudometric space via the function ϕ as described previously.

Let (X, d) be a second countable pseudometric space, and let $\{B_n\}_{n\in\mathbb{N}}$ be a countable basis for X - we claim that $\{\phi(B_n)\}_{n\in\mathbb{N}}$ forms a basis for the metric space $X' = \phi(X)$. For each $\phi(x) = x' \in X' = \phi(X)$ let V be a neighbourhood of $\phi(x)$. Since ϕ is both open and continuous, $\phi^{-1}(V)$ is an open set in X and is a neighbourhood of x. Then, because $\{B_n\}_{n\in\mathbb{N}}$ is a basis for X, there exists a basis element B such that $x \in B$. Then, since ϕ is an open mapping, $\phi(B)$ contains $f(x)$ and is open in X'. Hence, $\{\phi(B_n)\}_{n\in\mathbb{N}}$ is a countable basis for X', meaning X' is a second countable metric space.

Applying this fact to the proof from Theorem 3.1 gives the desired implication. □

Chapter 4

Relationship between effective and topologically principle

In this chapter, we begin by briefly discussing some preliminaries of topological groupoids, before introducing the two main properties of interest for the remainder of this paper - those of being topologically principle or effective. We also look at some examples of groupoids which highlight how these properties may arise separately or together, and that give an idea of what groupoids that have these properties might look like intuitively.

The following definitions can all be found in [9, Section 2.3]. A *groupoid* is a set G together with a distinguished subset $G^{(2)} \subseteq G \times G$, a multiplication mapping $(x, y) \mapsto xy$ from $G^{(2)} \to G$, and an inverse function $x \to x^{-1}$ from G to G, satisfying the following conditions:

(i) $(x^{-1})^{-1} = x$ for all $x \in G$;

(ii) if $(x, y), (y, z) \in G^{(2)}$, then (xy, z) and $(x, yz) \in G^{(2)}$, and $(xy)z = x(yz)$;

(iii) $(x, x^{-1}) \in G^{(2)}$ for all $x \in G$, and furthermore, for all $(x, y) \in G^{(2)}$, $x^{-1}(xy) = y$ and $(xy)y^{-1} = x$.

We think of $G^{(2)}$ as the set of ordered pairs for which the multiplication map is defined (i.e. the set of composable pairs).

There is also a subset $G^{(0)} \subseteq G$ called the *unit space* of G which is defined as $G^{(0)} = \{xx^{-1} : x \in G\}$. There exist range and source maps $r, s : G \to G^{(0)}$ such that

$$r(x) = xx^{-1} \quad \text{and} \quad s(x) = x^{-1}x$$

It can be helpful to note that composition of elements is "read" from right to left, much the same as composition as functions. This is why the range and source functions are defined as they are - the range function of an "arrow" is the object reached by following the arrow's inverse, and then the arrow itself.

A *topological groupoid* is a groupoid G along with a topology τ such that G is locally compact, the unit space $G^{(0)}$ is Hausdorff, and the maps r, s, $x \mapsto x^{-1}$, and $(x, y) \mapsto xy$ are continuous with respect to τ. We will be focusing on étale groupoids, which are topological groupoids such that the range map is a local homeomorphism. We also note that if G is an étale groupoid, then $G^{(0)}$ is open in G.

Let G be a topological groupoid. Define the sets $G_x = \{\gamma \in G : s(\gamma) = x\}$ and $G^x = \{\gamma \in G : r(\gamma) = x\}$, as well as $G_x^x = G_x \cup G^x$. Intuitively, G_x can be thought of as the set of all arrows pointing at x, and similarly for G^x, being the set of arrows "leaving" x. The isotropy subgroupoid of G, denoted $\text{Iso}(G)$, is defined as

$$\text{Iso}(G) = \bigcup_{x \in G^{(0)}} G_x^x.$$

This consists of all the elements $\gamma \in G$ such that $r(\gamma) = s(\gamma)$. This is indeed a subgroupoid, not a subgroup, as if $x, y \in G^{(0)}$ with $x \neq y$ then any element from G_x^x is not composable with any element from G_y^y, but any two elements from G_x^x are composable with one another. This is a straightforward example of a group bundle.

We may also need to compose subsets of a groupoid. If $U, V \subseteq G$, then we define

$$UV = \{\alpha\beta : \alpha \in U, \beta \in V, s(\alpha) = r(\beta)\}.$$

That is, UV is the set of composed pairs from $U \times V$ that are composable!

We say that G is *topologically principle* if the set $\{x \in G^{(0)} : G_x^x = \{x\}\}$ is dense in $G^{(0)}$ (if $G_x^x = \{x\}$ then we may say that x has *trivial isotropy*). We also say that G is *effective* if $\text{Int}(\text{Iso}(G)) = G^{(0)}$. A subset U of $G^{(0)}$ is called *invariant* if, for all $\gamma \in G$, $s(\gamma) \in U \implies r(\gamma) \in U$. An open set $B \subseteq G$ is called an *open bisection* of G if $r|B$ and $s|B$ are homeomorphisms onto open subsets of $G^{(0)}$. An important fact which we will use is that a groupoid is étale if and only if it has a basis of open bisections - in particular, étale groupoids have covers of open bisections.

In order to motivate the following results, let us consider some examples of groupoids which have neither, just one, or even both of the topologically principle and effective properties.

Example 4.1. Consider the trivial groupoid that has a single element and single morphism (the identity morphism) - let us say $G = \{a\}$. We can define the discrete topology on G, whereby we will have $\mathcal{T} = \{\{a\}, \emptyset\}$. Note that $G = G^{(0)} = \{a\}$. It is easy to see that G is topologically principle, since

$$\{x \in G^{(0)} : G_x^x = \{x\}\} = \{a\}$$

In addition, G is effective, since $\text{Iso}(G) = \{a\}$ and then $\text{Int}(\text{Iso}(G)) = \{a\} = G^{(0)}$.

Example 4.2. Take $G = G^{(0)} = [0,1]$ and $G^{(2)} = [0,1] \times [0,1]$ as a topological groupoid with the order topology whereby every element is composable with every other element. Then $\{x \in G^{(0)} : G_x^x = \{x\}\} = \emptyset$ and so G is not topologically principle. In addition, $\text{Iso}(G) = G$, and $\text{Int}(\text{Iso}(G)) = (0,1) \neq G^{(0)}$. Hence, G is not effective either.

Example 4.3. Let us define $X = (0,1) \times \mathbb{T}$, where $\mathbb{T}$ is the complex unit circle group (the group of complex numbers z such that $|z| = 1$). Define a continuous action of $\mathbb{R}$ on X by

$$t \cdot (s, e^{i\theta}) = (s, e^{i(\theta + 2st\pi)})$$

where $t \in \mathbb{R}$, $s \in (0,1)$ and $e^{i\theta} \in \mathbb{T}$. For example, if $t = 2$, $s = 1/2$ and our complex number is $e^{i\pi}$, then

$$2 \cdot (1/2, e^{\pi i}) = (1/2, e^{i(\pi + 2\pi)}) = (1/2, e^{\pi i})$$

We can see here that $(1/2, e^{\pi i})$ is a fixed point of $1/2$. Define G as the transformation-group groupoid $X \rtimes \mathbb{R}$. This is the groupoid associated to the group action of $\mathbb{R}$ on X defined by the following;

(i) $G^{(0)} = X = (0,1) \times \mathbb{T}$;

(ii) for two elements $x, y \in X$, the morphisms from x to y are the elements $t \in \mathbb{R}$ such that $t \cdot x = y$;

(iii) composition of morphisms is the binary operation of $\mathbb{R}$.

Take each $u = (s, e^{i\theta}) \in G^{(0)} = X$. Note that in this groupoid the source and range maps are defined by $s(t,(s,e^{i\theta}) = (s,e^{i\theta})$ and $r(t,(s,e^{i\theta}) = t \cdot (s,e^{i\theta})$. Let us consider, for each $u \in G^{(0)}$, the isotropy group G_u^u defined as usual by $G_u^u = \{\gamma \in G : s(\gamma) = r(\gamma) = u\}$. In this case, the criteria of $s(\gamma) = \gamma$ is satisfied by all elements of the action groupoid by definition, so it is only required further than $r(\gamma) = u$ - or in other words, $G_u^u = \{(u,t) \in X \times \mathbb{R} : r(u,t) = u\}$, whereby $r(u,t) = t \cdot u$. Hence, it can be seen that our isotropy group of u is $\{u\} \times \frac{1}{s}\mathbb{Z}$. In our previous example, the isotropy group of $(1/2, e^{\pi i})$ would be $\{(1/2, e^{\pi i})\} \times 2\mathbb{Z}$. This means that there are no points in $G^{(0)}$ with trivial isotropy. This means that our groupoid G is not topologically principle, since $\{x \in G^{(0)} : G_x^x = \{x\}\}$ is empty, so certainly not dense in $G^{(0)}$.

Fix an open set $U \subseteq G$. We want to show that $U - \text{Iso}(G) \neq \emptyset$. Since U is an open set, there exists $0 < a < b < 1$, $\theta \in (0, 2\pi)$, and $t \in \mathbb{R} - \{0\}$, such that $((a,b) \times \{e^{i\theta}\}) \times \{t\} \subseteq U$. Fix some $s \in (a,b)$. If $st \notin \mathbb{Z}$ then $((s,e^{i\theta}),t) \in U - \text{Iso}(G)$. So, suppose $st \in \mathbb{Z}$. Take some $\varepsilon \in (0, 1/t)$ such that $s + \varepsilon \in (a,b)$. Then,

$$st < (s+\varepsilon)t < st + 1$$

so $(s+\varepsilon)t \notin \mathbb{Z}$. Hence, $((s+\varepsilon), e^{i\theta}), t) \in U - \text{Iso}(G)$. [3]

Example 4.4. Consider the closed interval $[0,1]$ in the real line along with an additional point α. We assign a topology on this space given by the following basis;

$$\mathcal{B} = \{(a,b) : 0 < a < b \leq 1\} \cup \{[0,a) : 0 < a \leq 1\} \cup \\ \{(b,1] : 0 \leq b < 1\} \cup \{([0,a) - \{0\}) \cup \{\alpha\} : 0 \leq a \leq 1\}$$

We can further consider this as a topological groupoid G, where $G = [0,1] \cup \{\alpha\}$, $G^{(0)} = [0,1]$, and $G^{(2)} = \{(x,x) : x \in [0,1]\} \cup \{(\alpha,\alpha), (0,\alpha), (\alpha,0)\}$. That is, each real number in the unit interval is only composable with itself, except for 0 which is composable with our extra element α. Intuitively, this space is the unit interval with two elements which act as 0.

We can observe that this space is non-Hausdorff. Suppose it is Hausdorff. Then for the points 0 and α there exist neighbourhoods U_1 and U_2, respectively, of both which are disjoint. Hence, there exist basis elements B_1 and B_2, respectively, such that $0 \in B_1 \subseteq U_1$ and $\alpha \in B_2 \subseteq U_2$. If 0 is in a basis element, then it is of the form $[0,a)$ for some $a \in (0,1]$. Similarly, if α is contained in a basis element, then it must be of the form $([0,b) - \{0\}) \cup \{\alpha\}$. Clearly, these sets have non-empty intersection (they both contain the real number $\frac{\min\{a,b\}}{2}$). Thus, the neighbourhoods U_1 and U_2 also have non-empty intersection. This is a contradiction. Hence, it must be that our space is non-Hausdorff.

Let us now consider whether this space is topologically principle and/or effective. Firstly, we will look at the set $\{x \in G^{(0)} : G_x^x = \{x\}\}$ i.e. the set of units with trivial isotropy. By definition of this topological groupoid, this is the set $(0,1]$ as every unit is composable only with itself except 0. Clearly, $(0,1]$ is dense in $[0,1]$, meaning this space is topologically principle. Now, let us take $\text{Int}(\text{Iso}(G))$. We have $\text{Iso}(G) = G$, and so $\text{Int}(\text{Iso}(G)) = \text{Int}(G)$ which is certainly non-empty. Hence, G is not effective.

Lemma 4.5. *Let G be an étale, Hausdorff groupoid. Then G is topologically principle iff every open subset of $G^{(0)}$ contains a point with trivial isotropy.*

Proof. Clearly, if G is topologically principle, then the set of points with trivial isotropy is dense in $G^{(0)}$, and so for every point x, every neighbourhood of x contains a point with trivial isotropy. Hence, every open subset of $G^{(0)}$ contains a point with trivial isotropy.

Conversely, suppose that every open subset of $G^{(0)}$ contains such a point. For each $x \in G^{(0)}$, if $G_x^x = \{x\}$, then the dense condition holds trivially. If not, then every neighbourhood of x contains a point with trivial isotropy, and the dense condition is satisfied. In either case, we can see that G is topologically principle. □

Lemma 4.6. *Let G be a locally compact, Hausdorff, étale groupoid. Then the following are equivalent:*

(a) The interior of the isotropy subgroupoid of G is $G^{(0)}$*;*

(b) For every non-empty bisection $B \subseteq G - G^{(0)}$*, there exists* $\gamma \in B$ *such that* $s(\gamma) \neq r(\gamma)$*.*

Proof. To see that $(a) \implies (b)$, we can see first that since G is étale, G is clopen, and so for any subset $S \subseteq G$, we can write

$$\mathrm{Int}(S) = \mathrm{Int}(S \cap G^{(0)}) \cup \mathrm{Int}(S - G^{(0)})$$

Taking $S = \mathrm{Iso}(G) - G^{(0)}$, we can see that

$$\begin{aligned}\mathrm{Int}(\mathrm{Iso}(G) - G^{(0)}) &= \mathrm{Int}(\mathrm{Iso}(G)) \cup \mathrm{Int}(\mathrm{Iso}(G) - G^{(0)}) \\ &= G^{(0)} \cup \mathrm{Int}(\mathrm{Iso}(G) - G^{(0)}) \\ &= \emptyset\end{aligned}$$

This means $\mathrm{Iso}(G) - G^{(0)}$ has empty interior. Now, let B be an open bisection. Suppose for all $\gamma \in B$, $s(\gamma) = r(\gamma)$. This would mean $B \subseteq \mathrm{Iso}(G)$, and since B is open, $B \subseteq \mathrm{Int}(\mathrm{Iso}(G))$. By assumption, $B \not\subseteq G^{(0)}$, and so we also have $B \subseteq \mathrm{Int}(\mathrm{Iso}(G) - G^{(0)})$. This contradicts the fact that the interior of this set is empty. Hence, there exists some $\gamma \in B$ such that $s(\gamma) \neq r(\gamma)$.

Conversely, we observe that

$$\mathrm{Int}(\mathrm{Iso}(G) - G^{(0)}) = \bigcup_n U_n$$

where U_n is every open set in $\mathrm{Iso}(G) - G^{(0)}$. Since the open bisections form a basis, for each n, U_n is either empty or contains a non-empty open bisection B. Since $\mathrm{Iso}(G) - G^{(0)} \subseteq G - G^{(0)}$, this basis element B has some element γ such that $r(\gamma) \neq s(\gamma)$. But then B cannot be contained in $\mathrm{Iso}(G)$, and so U_n must be empty. This holds for each set U_n, and so their union (and hence the interior of the given set) is empty. Using the equivalence given earlier, this then means that the interior of $\mathrm{Iso}(G)$ is $G^{(0)}$. [3, Lemma 3.1] □

Theorem 4.7. Let G be a locally compact, Hausdorff, second countable, étale groupoid. If G is effective, then G is topologically principle.

Proof. Let G be a second countable étale groupoid, and suppose that G satisfies condition (b) in the above lemma (is effective). Define U to be the interior of the set of units with non-trivial isotropy - we wish to show that U is empty. If $s(\gamma) \in U$, then there exists an element x such that $s(x) = r(x) = u$, but $x \neq u$. Then, $r(\gamma x) = r(\gamma)$ and $s(\gamma x \gamma^{-1}) = s(\gamma^{-1}) = r(\gamma)$ (note that these compositions are all defined since $r(x) = s(x) = r(\gamma) = s(\gamma)$. Hence, $r(\gamma)$ has non-trivial isotropy. Then, $r(G_U)$ is open and consists of points with non-trivial isotropy so is contained in U. We can see that U is an open invariant set. Define $H := G_U$, which we can see is a second countable, locally compact, Hausdorff, étale groupoid in which every unit has non-trivial isotropy. We will now show that $H^{(0)} = U$.

In one direction, let $u \in U$, and we show $u \in H^{(0)}$. By definition, there exists an element $x \in G$ such that $r(x) = s(x) = u$. Hence, $u \in G^{(0)} \cap G_U$ and so $U \in H^{(0)}$.

In the other direction, suppose $x \in H^{(0)}$. Then, $s(x) = r(x) = x$, but because $H^{(0)} \subseteq G_U$, $x \in U$. Therefore, we have $H^{(0)} = U$.

Consider the set $H - U = H - H^{(0)}$, and take an open bisection $B \in H - U$. Define $B_{\text{Iso}} := \{\gamma \in B : s(\gamma) = r(\gamma)\}$. We claim that $r(B_{\text{Iso}}) = s(B_{\text{Iso}}) = \{r(\gamma) : \gamma \in B, r(\gamma) = s(\gamma)\}$ is nowhere-dense i.e. every open set in its closure is empty. So, we take an open set $V \subseteq \overline{r(B_{\text{Iso}})}$. We define $VB = \{\alpha\beta : \alpha \in V, \beta \in B, s(\alpha) = r(\gamma)\}$, which is the product of two open sets, and so is itself open - in particular, it is an open subset of B which is a bisection, meaning it itself is a bisection. Let $\gamma \in VB$. To see that $r(\gamma) \in V$, suppose $\gamma = \alpha\beta$ where $\alpha \in V$ and $\beta \in B$. Then $r(\gamma) = r(\alpha\beta) = r(\alpha) = \alpha$ since V consists entirely of units. Since $r(\gamma) \in V \subset \overline{r(B_{\text{Iso}})}$, we can write $r(\gamma) = \lim_{n\to\infty} v_n$ for some sequence $(v_n) \subseteq r(B_{\text{Iso}})$. Now consider the set $v_n B$ for each n. Since B is a bisection, and the range map is injective, this set has a unique element - call this γ_n. These form a sequence (γ_n). Since $r|_B$ is a homeomorphism, we can see that $\gamma = \lim_{n\to\infty} r^{-1}(v_n) = \lim_{n\to\infty} \gamma_n$. Since s, r is continuous, this gives us $s(\gamma) = \lim_{n\to\infty} s(\gamma_n) = \lim_{n\to\infty} r(\gamma_n) = r(\gamma)$. This means that there is an open bisection VB, and for every $\gamma \in VB$, $s(\gamma) = r(\gamma)$. By the second definition of effectiveness in the previous lemma, this implies VB is empty, and hence V is empty. Thus, $r(B_{\text{Iso}})$ is nowhere-dense.

Since H is second-countable, and $H - U$ is open, there exists a countable collection $\mathcal{B}$ of open bisections in H such that $H - U = \bigcup \mathcal{B}$. We now claim that $\bigcup_{B\in\mathcal{B}} r(B_{\text{Iso}}) = U$. Let $x \in \bigcup_{B\in\mathcal{B}} r(B_{\text{Iso}})$. Then, $x = r(\gamma)$ for some $\gamma \in B_{\text{Iso}}$, meaning $r(\gamma) = s(\gamma)$ and $\gamma \in B$. But, $B \subseteq H - U \subseteq H = G_U$. Hence, $s(\gamma) = r(\gamma) = x \in U$. By the previous lemma, H

In the other direction, let $x \in U$. Then, by definition of U, there exists an element $\gamma \in G_x^x$ with $\gamma \neq x$. Since $s(\gamma) = r(\gamma) = x$, $\gamma \in H = G_U$, but γ is not a unit of H (since its range and source are not itself) so $\gamma \in H - U$. Hence, there exists $B \in \mathcal{B}$ with $\gamma \in B$. In fact, $\gamma \in B_{\text{Iso}}$. So, $x \in r(B_{\text{Iso}}) \subseteq \bigcup B \in \mathcal{B} r(B_{\text{Iso}})$. This gives us $\bigcup_{B\in\mathcal{B}} r(B_{\text{Iso}}) = U$. So, U is a locally compact, Hausdorff space which can be written as the countable union of nowhere-dense sets. Thus, the Baire Category Theorem [6, Theorem 6.34] states that U is empty. □

Theorem 4.8. Let G be a locally compact, second countable, étale groupoid such that $G^{(0)}$ is Baire. If G is effective, then G is topologically principle. [8, Prop. 3.6]

Proof. Let us defined Y as the set of points in $G^{(0)}$ with trivial isotropy, and take Z to be the complement in the unit space; $Z\ G^{(0)} - Y$. We wish to show that Y is dense in $G^{(0)}$. Assume G is effective. Let us take a cover of G comprised of a countable collection of open bisections (S_n). Such a cover is guaranteed given that G is étale. Define, for each n, $A_n = r(S_n \cap \text{Iso}(G))$ - that is, A_n is the set of range objects of those elements in the open bisection S_n with trivial isotropy. Now, consider the sets $Y_n = \text{Int}(A_n) \cup \text{Ext}(A_n)$. Since both the interior and exterior of A_n are open, Y_n is also open. Furthermore, we can see that each Y_n is dense in $G^{(0)}$ - for each unit x, either it is a member of either the interior or exterior of A_n or a member of the boundary of A_n, in which case it is a limit point. Since $G^{(0)}$ is Baire, we know that the countable intersection of dense subsets is itself dense, meaning $\bigcap_n Y_n$ is dense in $G^{(0)}$. We aim to show that $\bigcap_n Y_n \subseteq Y$. This would imply Y is dense in $G^{(0)}$, since $\bigcap_n Y_n \subseteq Y \subseteq G^{(0)}$, and $\bigcap_n Y_n$ is dense in $G^{(0)}$.

Let $x \in \bigcap_n Y_n$, and $\gamma \in G_x^x$. There exists n such that $x \in S_n$ (since the open bisections S_n cover G). Then, $\gamma \in S_n \cap \text{Iso}(G)$, and furthermore, $x = r(\gamma) = s(\gamma) \in A_n = r(S_n \cap \text{Iso}(G))$. In addition, since $x \in A_n$ and $x \in Y_n$ (so in either the interior or exterior of A_n), we have $x \in \text{Int}A_n$. Let V be an open set such that $x \in V \subseteq A_n$. Since r is a bijection from $S_n \cap \text{Iso}(G)$ onto A_n (since $S_n \cap \text{Iso}(G) \subseteq S_n$, and r is a homeomorphism on S_n being an open bisection), the open set VS_n is contained in $\text{Iso}(G)$. Hence, $VS_n \subseteq G^{(0)}$ and $\gamma = xS_n \in G^{(0)}$ so $x \in Y$. Hence, $\bigcap_n Y_n \subseteq Y$, giving us the desired result that G is effective. □

Lemma 4.9. *Let G be an étale, Hausdorff groupoid. Then, the following are equivalent;*

(1) $Int(Iso(G)) = G^{(0)}$,

(2) $Iso(G) - G^{(0)}$ has empty interior.

If G is non-Hausdorff, then we only have $(1) \implies (2)$.

Proof. Let G be Hausdorff. We show that conditions 1 and 2 are equivalent. Suppose that $\text{Int}(\text{Iso}(G)) = G^{(0)}$, and toward a contradiction, suppose $x \in \text{Int}(\text{Iso}(G) - G^{(0)})$ for some element $x \in G$. So, there exists a neighbourhood U of x contained in $\text{Iso}(G)$ admitting an empty intersection with $G^{(0)}$. Hence, x is in the interior of $\text{Iso}(G)$, and so by assumption, is also in $G^{(0)}$. This contradicts U having empty intersection with $G^{(0)}$. This gives us $(1) \implies (2)$.

In the other direction, suppose that $\text{Iso}(G) - G^{(0)}$ has empty interior, and let $x \in \text{Int}(\text{Iso}(G))$. We first aim to show that $x \in G^{(0)}$. Since $x \in \text{Int}(\text{Iso}(G))$, there exists an open set U containing x such that $U \subseteq \text{Iso}(G)$. Note that U must intersect $G^{(0)}$ at some point. If this weren't the case, then we would have $x \in \text{Int}(\text{Iso}(G) - G^{(0)})$, but we know this to be empty. Consider the set $U - G^{(0)}$, which is open (being an open set minus a closed set). If $U - G^{(0)}$ is non-empty, it would imply that the interior of $\text{Iso}(G) - G^{(0)}$ is also non-empty. So it must be that $U - G^{(0)}$ is empty i.e. $U \subseteq G^{(0)}$. Hence, $x \in G^{(0)}$ giving us $\text{Int}(\text{Iso}(G)) \subseteq G^{(0)}$.

Now, for the other inclusion, it is easy to see that since G is étale, $G^{(0)}$ is both contained in $\text{Iso}(G)$ and open, and so is contained in the interior of $\text{Iso}(G)$. Therefore, we can conclude that $\text{Int}(\text{Iso}(G)) = G^{(0)}$. □

We will take the first condition above to be the usual definition of effective, and call the second condition *weakly effective*.

Theorem 4.10. Let G be a locally compact, étale groupoid such that

(i) G has a countable cover of open bisections, and

(ii) $G^{(0)}$ is Baire.

If G is weakly effective, then G is topologically principle.

In Clark and Brown's similar formulation of the above theorem [2], they employed the use of neighbourhood bisections, which are weaker than open bisections (i.e. every neighbourhood bisection is an open bisection). These are generalizations in the sense that neighbourhood bisections can be closed sets (but, for instance, require the interior to be an open bisection), while open bisections are required to be open. Hence, the following lemma follows easily.

Lemma 4.11. *Let G be a topological groupoid such that $G^{(0)}$ is open in G, and let $B \subseteq G$ be an open bisection. If $D \subseteq B$ is closed in B then $r(D)$ is closed in $r(B)$.*

Proof. Since B is an open bisection, $r|_B$ is a homeomorphism and so is a closed map. Hence if D is closed in B, $r(D)$ is closed in $r(B)$. □

Lemma 4.12. *Suppose that G is a topological groupoid whereby $G^{(0)}$ is open in G, B is an open bisection and G is weakly effective. Then both $r(Iso(B) - G^{(0)})$ and its closure $\overline{r(Iso(B) - G^{(0)})}$ have empty interior.*

Proof. To show that $r(\text{Iso}(B) - G^{(0)})$ has empty interior, toward a contradiction, suppose there is some open set W such that $W \subseteq r(\text{Iso}(B) - G^{(0)})$. Hence $W \cap r(B)$ is non-empty, and since by definition we have $B \subseteq \overline{\text{Int}(B)}$, this implies $W \cap r(\overline{\text{Int}(B)}) \neq \emptyset$ as well. Then, because $\overline{r(\text{Int}(B))} \subseteq r(\overline{\text{Int}(B)})$, this implies that $W \cap \overline{r(\text{Int}(B))}$ is non-empty. Hence $W \cap r(\text{Int}(B))$ is an open set contained in $W \cap \overline{r(\text{Int}(B))}$ (being the union of two open sets) and so contained in $G^{(0)}$. We consider the range map r restricted to the open set $\text{Int}(B)$, which is a homeomorphism onto its image (since they are homeomorphisms when restricted to the open bisection B and hence to the subset $\text{Int}(B)$). Hence, $r^{-1}(W \cap r(\text{Int}(B)))$ is a non-empty open subset of G. Since $W \subseteq r(\text{Iso}(B) - G^{(0)})$, we have that

$$(r|_{\text{Int}(B)}(W \cap r(\text{Int}(B))) \subseteq \text{Iso}(B) - G^{(0)} \subseteq \text{Iso}(G) - G^{(0)}.$$

But $r|_{\text{Int}(B)}(W \cap r(\text{Int}(B)))$ is an open set as we've seen, and it is contained in $\text{Iso}(G) - G^{(0)}$, which contradicts the fact that G is weakly effective. Hence, $r(\text{Iso}(B) - G^{(0)})$ has empty interior.

Now, we show that the closure of the above set also has empty interior. We use a similar strategy - assume there exists a non-empty open set $V \subseteq \overline{r(\text{Iso}(B) - G^{(0)})}$. Since the intersection of open sets is open, $V \cap r(B)$ is open. In addition, $V \cap r(B) \subseteq \overline{r(\text{Iso}(B) - G^{(0)})} \cap r(B)$. We now wish to show that $\overline{r(\text{Iso}(B) - G^{(0)})} \cap r(B) = r(\text{Iso}(B) - G^{(0)})$. We know that $\text{Iso}(B)$ is closed in B, and $G^{(0)}$ is open, $\text{Iso}(B) - G^{(0)}$ is closed in B. We use Lemma 4.2 - note that the condition that $B - \text{Int}(B) \subseteq D$ is satisfied vacuously as B is an open bisection, and so $B = \text{Int}(B)$. Hence, the lemma gives us that $r(\text{Iso}(B) - G^{(0)})$ is closed $r(B)$. Hence, $\overline{r(\text{Iso}(B) - G^{(0)})} = r(\text{Iso}(B) - G^{(0)}) \cap r(B)$. From our assumption this gives us $V \cap r(B) \subseteq r(\text{Iso}(B) - G^{(0)})$, contradicting the first part of this lemma. [2] □

Proof of Theorem 4.10. Since G is étale, it has a countable cover of open bisections. We denote this cover $\{B_n\}$. Suppose that G is weakly effective, such that $\text{Iso}(G) - G^{(0)}$ has empty interior. We define $C_n = \overline{r(\text{Iso}(B_n) - G^{(0)})}$. That is, each C_n is the closure of the set of objects that are ranges of elements in the set $\text{Iso}(B_n) - G^{(0)}$. By the previous lemma, this set has empty interior for each n. Each set is also closed in $G^{(0)}$, while $G^{(0)}$ is open in G, and so they also have empty interior in $G^{(0)}$. Since $G^{(0)}$ is Baire, $C = \bigcup_n C_n$ has empty interior in $G^{(0)}$. I lastly claim that C is exactly the units with non-trivial isotropy. If x is an object with non-trivial isotropy, there is some element γ with $r(\gamma) = s(\gamma) = x$. Let n be such that $\gamma \in B_n$. Then $\gamma \in \text{Iso}(B_n) - G^{(0)}$ and so $x \in r(\text{Iso}(B_n) - G^{(0)}) \subseteq \overline{r(\text{Iso}(B_n) - G^{(0)})} \subseteq C$. Hence C is the set of objects with non-trivial isotropy and has empty interior, and so G is topologically principle. □

Chapter 5

Baire Spaces and Choice

5.1 Countable Choice

Theorem 5.1. The following are equivalent:

(1) let G be an étale groupoid that has a countable cover of open bisections, such that $G^{(0)}$ is a second-countable complete metric space. If G is weakly effective, it is topologically principle.

(2) the Axiom of Countable Choice.

Before proving the above theorem, we will proceed with the construction of a topological groupoid from a second-countable metric space. Let X be a second-countable metric space, and let $(A_n)_{n=1}^{\infty} \subset X$ be a countable sequence of closed subsets, such that each A_i has empty interior. Define $A = \bigcup A_i$. We wish to construct a groupoid G such that $G^{(0)} = X$. We add elements by, for each $a \in A$, adding a point γ_a to G such that $r(\gamma_a) = s(\gamma_a) = a$. Hence, $G = G^{(0)} \cup \{\gamma_a : a \in A\}$. We define the composable pairs of elements as

$$G^{(2)} = \{(x, x) : x \in X\} \cup \{(\gamma_a, a), (a, \gamma_a), (\gamma_a, \gamma_a) : a \in A\}.$$

For each $x \in G^{(0)}$, clearly $x = x^{-1}$, and this self-inverse property also holds for every additional point γ_a. Furthermore, we see that $\text{Iso}(G) = G$, and so G can be considered a group bundle where Γ_x is the trivial group for $x \in G^{(0)}$, and Γ_{γ_a} is isomorphic to $\mathbb{Z}_2$ for $a \in A$. These facts demonstrate that G is indeed a well-defined groupoid.

Suppose X has a basis $\{B_n\}_{n=1}^{\infty}$. We construct a topology on G defined by the following basis

$$\{D_n\} = \{B_n\} \cup \{C_{n,A}\}$$

where for each n,

$$C_{n,A} \coloneqq (B_n - A) \cup \{\gamma_a : a \in B_n \cap A\}.$$

Intuitively, each basis element D_n of G is formed by removing every element $a \in A$ from B_n and replacing these with their respective γ_a. Note that since C need not be countable, $\{D_n\}$ may not be a countable collection.

Lemma 5.2. *Suppose X has a basis $\{B_n\}_{n=1}^{\infty}$. We construct a topology on G generated by the collection of sets*

$$\{D_n\} = \{B_n\} \cup \{C_{n,x}\},$$

where for each every n and every $x \in B_n \cap A$,

$$C_{n,x} := (B_n - \{x\}) \cup \{\gamma_x\}.$$

That is, each $C_{n,x}$ is formed by picking a single $x \in B_n \cap A$ and replacing it with the corresponding γ_x. Then, this collection satisfies the conditions for a topological basis on G.

Proof. First, take any $x \in G$. We wish to find a basis element D_n of G such that $x \in D_n$. If $x \in G^{(0)} = X$, then we know $x \in B_x$ for some basis element B_x of X. Hence, x is in the corresponding basis element D_x of G. On the other hand, suppose $x \in G - G^{(0)}$ - say $x = \gamma_a$ for some $a \in A$. There exists a basis element B_a containing a, and so the new basis element $C_{a,A}$ formed by replacing the point a with γ_a contains our point x.

Now, again taking $x \in G$, suppose $x \in D_1 \cap D_2$ for two basis elements D_1, D_2 of G. We wish to show there exists another basis element $D_3 \subseteq D_1 \cap D_2$ containing x. Once again we use a case by case approach. Suppose $x \in G^{(0)}$ and $x \in B_1 \cap B_2$. Thus, there is a third basis element B_3 of X such that $x \in B_3 \subseteq B_1 \cap B_2$. Next, suppose $x \in C_{1,A} \cap C_{2,A}$, where $C_{1,A} = (B_1 - \{x\}) \cup \{\gamma_x\}$ and $C_{2,A} = (B_2 - \{y\}) \cup \{\gamma_y\}$, where $x \in B_1 \cap A$ and $y \in B_2 \cap A$. Recall that these are basis elements of G not contained in the unit space. We replace $C_{1,A}, C_{2,A}$ with their corresponding basis elements in $G^{(0)}$, B_1, B_2 formed by removing the non-unit elements γ_x and γ_y and replacing them with their respective elements $x, y \in A$. As before, since the sets $\{B_n\}$ form a basis for X, there exists a $B_3 \subseteq B_1 \cap B_2$ containing x. Since $X - A$ is open, there exists an open basis element B_4 containing x which is contained within $B_3 - A$. Since, for each n,

$$B_n - \{x\} \subseteq C_{n,A},$$

for some $x \in B_n \cap A$, we then see that

$$B_4 \subseteq B_3 - \{z_1\} \subseteq (B_1 \cap B_2) - \{z_2\} \subseteq C_{1,A} \cap C_{2,A}$$

where $z_1 \in B_3 \cap A$ and $z_2 \in (B_1 \cap B_2) \cap A$. This same argument applies if we have $x \in B_1 \cap C_{2,A}$. Hence, we see this satisfies our criteria as a basis for G. $\square$

Before we can show G is an étale groupoid, we will prove it is a topological groupoid. We have seen the basis above defines a topology on G, but it remains to show that G is locally compact and the inverse and multiplication maps are continuous.

Lemma 5.3. *As a topological groupoid, G is locally compact, and the inverse and multiplication maps are continuous.*

Proof. Since every element of G is its own inverse, the inverse map is the identity map and so is trivially continuous. Now, consider the multiplication map $m : G^{(2)} \to G$. Let B_n be a basis element of G contained in the unit space. Then, the inverse image of B_n under multiplication is defined as

$$\{(\alpha, \beta) \in G^{(2)} : s(\alpha) = r(\beta), \alpha\beta \in B_n\}.$$

The topology on $G^{(2)}$ is the subspace topology inherited from the product space $G \times G$. Since each unit is only composable with itself, we have the inverse image of B_n under multiplication as being

$$m^{-1}(B_n) = \{(x, x) : x \in B_n\} \cup \{(\gamma_x, \gamma_x) : x \in B_n \cap A\}.$$

If we take a basis element instead to be $C_{n,A}$ for some n, then the inverse image under multiplication is

$$m^{-1}(C_{n,A}) = \{(\gamma_x, \gamma_x), (\gamma_x, x), (x, \gamma_x) : x \in C_{n,A} \cap A\} \cup \{(x, x) : x \in C_{n,A} \cap X\}$$

Both of these are open in G. To see these, we consider a number of cases. If our basis element is B_n, let $\alpha \in m^{-1}(B_n)$. If $\alpha = (x, x)$ for some $x \in B_n$, then

$$(x, x) \in (B_n \times B_n) \cap G^{(2)} \subseteq m^{-1}(C_{n,A})$$

which is a basis element of $G^{(2)}$ and is contained in $m^{-1}(B_n)$. Alternatively if $\alpha = (\gamma_x, \gamma_x)$ for some $x \in B_n \cap A$, then we can see

$$(\gamma_x, \gamma_x) \in (C_{n,A} \times C_{n,A}) \cap G^{(2)} \subseteq m^{-1}(C_{n,A}).$$

Hence, in either case, $m^{-1}(B_n)$ is open.

Now, let $\alpha \in m^{-1}(C_{n,A})$ - say $\alpha = (\gamma_x, \gamma_x)$ for some $x \in C_{n,A} \cap A$. Then,

$$(\gamma_x, \gamma_x) \in (C_{n,A} \times C_{n,A}) \cap G^{(2)}.$$

Since $C_{n,A} \times C_{n,A}$ is a basis element of $G \times G$, the set above is a basis element of $G^{(2)}$ containing (γ_x, γ_x) and contained in the inverse image of $C_{n,A}$. Now, without loss of generality, if we have $\alpha = (x, \gamma_x)$ or (γ_x, x) for some $x \in C_{n,A} \cap A$, then

$$(x, \gamma_x) \in (B_n \times C_{n,A}) \cap G^{(2)} \subseteq m^{-1}(C_{n,A}).$$

Lastly, if $\alpha = (x, x)$ for some $x \in C_{n,A}$ then

$$(x, x) \in (B_n \times B_n) \cap G^{(2)} \subseteq m^{-1}(C_{n,A}).$$

In each of these cases, there exists a basis element containing our point, which itself is contained in the inverse image, meaning the inverse image is open. Hence, multiplication is continuous. □

Lemma 5.4. *The groupoid G is an étale groupoid. That is, the range map is locally homeomorphic as a map from G to $G^{(0)}$.*

Proof. Henceforth we note that the range map on the unit space is the identity map so is trivially a homeomorphism. So, when we take $\alpha \in G$ we only consider when $\alpha \in G - G^{(0)}$ i.e. $\alpha = \gamma_a$ for some $a \in A$. Take $\alpha \in G$ and suppose $\alpha = \gamma_a$ for some point $a \in A$, whereby $r(\alpha) = a$, and fix a basis element C_α containing α, which will serve as our neighbourhood on which r is a homeomorphism. Clearly r is surjective onto $r(C_\alpha)$, as $r(x) = x$ for every $x \in G^{(0)}$, and $r(C_\alpha) \subseteq G^{(0)}$. Now, suppose $r(x) = r(y)$ for two points $x, y \in C_\alpha$. If neither point is a γ_a then this can only be the case if $x = y$ since $r(x) = x = y = r(y)$. If we have $x = \gamma_a$ and $y = \gamma_b$ for points $a, b \in A$, then we only have $r(x) = r(y)$ if $x = y$ since there is a unique γ_a associated to each point in a. Lastly, without loss of generality suppose $x \in G^{(0)}$ and $y = \gamma_a$ for some $a \in A$. Then, if $r(x) = r(y)$, we have $r(x) = x = a = r(y)$ which is impossible since γ_a and a cannot be in the same basis element. Hence, we see r is injective on C_α. Note - r only has to be bijective on our fixed basis element C_α, not every neighbourhood of α.

We can see that r is locally bijective, and it remains to show that it is open and continuous as a function $r : C_\alpha \to r(C_\alpha)$. Since r is the identity map on every point except the added γ elements, but takes each γ_a to a, we see that

$$r(C_\alpha) = C_\alpha - \{\gamma_b : \gamma_b \in C_\alpha\} \cup \{b : \gamma_b \in C_\alpha\}$$

which, by the definition of our basis, can be seen to be equal to a basis element (namely, one of the original basis elements of X which is carried over into the topology for G) which is open. Hence, r is open.

Now, we wish to show that r is continuous. It suffices to show that if $B_i \subseteq G^{(0)}$ is an open basis element of G, then $r^{-1}(B_i)$ is open in G. Note that $r^{-1}(B_i) = \{\gamma \in G : r(\gamma) \in B_i\}$. If B_i intersects A, then

$$r^{-1}(B_i) = B_i \cup C_{i,A}.$$

Otherwise, if B_i does not intersect A, then $r^{-1}(B_i) = B_i$. In the first instance we have the union of two open sets which is open, and in the second case clearly B_i is open. It follows from the fact that r is locally bijective, open and globally continuous that r^{-1} is also continuous. □

Lemma 5.5. *The groupoid G is not effective.*

Proof. We know that $\mathrm{Iso}(G) = G$, and since G is open in itself, $\mathrm{Int}(G) = G$. But, $G \neq G^{(0)}$. □

Lemma 5.6. *The groupoid G is always weakly effective.*

Proof. To see this, observe that $\mathrm{Iso}(G) - G^{(0)} = G - G^{(0)}$. Notice that since $G^{(0)}$ is a metric space, it is Hausdorff, and so all singletons are closed. Hence, there exists no singleton $\{a\} \in A$ such that $B_n = \{a\}$ for some basis element B_n. Hence, every basis element D_n of G intersects the unit space. This implies that $G - G^{(0)}$ contains no basis elements, and so has empty interior. □

Proof of Theorem 5.1. (2) $\implies$ (1): Assume that the axiom of countable choice holds. By [4, 0.15], this is equivalent to the assertion that every second-countable complete pseudometric space is Baire. Then, by Corollary 3.8, this is equivalent to the statement that every second-countable complete metric space is Baire. Since $G^{(0)}$ is a second-countable complete metric space, it is therefore Baire. Then, by Theorem 4.10, since G is weakly effective, it is topologically principle.

(1) $\implies$ (2): Since G is weakly effective, by the hypothesis of Theorem 5.1, G is also topologically principle. Recall that this means that the set of units with non-trivial isotropy has empty interior in $G^{(0)}$. In G, the set of units with non-trivial isotropy is exactly A. Hence, A has empty interior. Furthermore, since A is the union of a sequence of sets with empty interior, this implies that X is Baire, giving us our result that every complete metric space is Baire. By Theorem 3.1, this implies that every complete pseudometric space is Baire (which certainly means every complete second-countable pseudometric space is Baire). Finally, by [4, Theorem 0.15], we have the Axiom of Countable Choice. □

By proving this equivalence, we can add a new equivalent condition to Theorem 0.15 [4].

Corollary 5.7. The following are equivalent:

(i) every totally bounded complete pseudometric space is Baire,

(ii) every second countable complete pseudometric space is Baire,

(iii) every second countable complete metric space is Baire,

(iv) the Axiom of Countable Choice,

(v) let G be an étale groupoid that has a countable cover of open bisections, such that $G^{(0)}$ is a second-countable complete metric space. If G is weakly effective, it is topologically principle.

5.2 Dependent Choice

We now consider a similar theorem, where we don't require the unit space of our groupoid to be second-countable.

Theorem 5.8. The following are equivalent:

(1) let G be an étale groupoid with a countable cover of open bisections, such that $G^{(0)}$ is a complete metric space. If G is weakly effective, it is topologically principle.

(2) the Axiom of Dependent Choice.

Proof. (1) $\implies$ (2): Let X be a complete metric space, and construct a groupoid G on X as in the proof of the previous theorem. We can construct a basis on G in an identical manner, providing us with a topological groupoid. In addition, G is étale - the proofs of the relevant lemmas hold without the assumption of second-countability. In addition, we can further assert that G is never effective, but always weakly effective. Finally, the proof of the previous theorem again holds without the assumption of second-countability, giving us that X is Baire. Hence, by [4, Theorem 0.16], the Axiom of Dependent Choice holds.

(2) $\implies$ (1): Suppose that Dependent choice holds, and let G be a weakly effective étale groupoid such that $G^{(0)}$ is a complete metric space. By [4, Theorem 0.16], $G^{(0)}$ is Baire. Hence, by Theorem 4.10, we have that if G is weakly effective, it is topologically principle. □

By proving this equivalence, as before, we are able to append an new equivalent condition to Theorem 0.16 [4].

Corollary 5.9. The following are equivalent:

(i) every complete pseudometric space is Baire,

(ii) every complete metric space is Baire,

(iii) for every discrete space X, the space $X^{\mathbb{N}}$ is Baire,

(iv) the Axiom of Dependent Choice,

(v) let G be an étale groupoid with a countable cover of open bisections, such that $G^{(0)}$ is a complete metric space. If G is weakly effective, it is topologically principle.

5.3 Possible Further Results

In this chapter we have determined properties of topological groupoids that are equivalent to both the Axiom of Countable Choice, and the Axiom of Dependent Choice. These are the most useful in that they are equivalent to commonly encountered classes of metric spaces (i.e. those that are complete and Baire - and possibly second-countable).

One might wonder whether there is a similar equivalence we can form for the Axiom of Dependent Multiple Choice. By [4, Theorem 0.17], we know that dependent multiple choice is equivalent to every compact Hausdorff space being Baire. If we let X be a compact Hausdorff space, and use a similar construction to build a groupoid G on X such that $G^{(0)} = X$, then can we draw similar conclusions as in the previous theorems? One might think we can use this identical groupoid construction, and our results hold for all the properties for G we have shown to hold, thus implying that since G is weakly effective, and hence topologically principle, giving us that the unit space of G is Baire.

Conjecture 5.10. The following are equivalent:

(1) let G be an étale groupoid such that $G^{(0)}$ is a compact Hausdorff space. If G is weakly effective, it is topologically principle.

(2) the Axiom of Dependent Multiple Choice.

Chapter 6

Conclusions

We began this paper by proving that if X is a complete pseudometric space, we can use the process of metric identification to turn X into a complete metric space such that the property of being Baire is preserved. We used this to show, via argument by contradiction, that if every complete pseudometric space is Baire, then every complete metric space is Baire. It then followed that since every metric space is a pseudometric space that the above result is an equivalence.

We then introduced the notion of topological groupoids, and discussed some important algebraic and topological properties of these structures, notably those of being topologically principle and weakly effective. We then determined that in certain classes of toplogical groupoids, having just one of these properties implies having the other. The relationship between these two properties allowed us to prove that if we have a complete, second-countable metric space X, we can construct a groupoid G with X as its unit space in such a way that G is always weakly effective, which implied it is topologically principle, which in turn implied X is Baire, which we know by [4] to be equivalent to the Axiom of Countable Choice. We extended this result to hold for the Axiom of Dependent Choice when the requirement of being second-countable is removed.

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